WEIGHTED NORM INEQUALITIES, OFF-DIAGONAL ESTIMATES AND ELLIPTIC OPERATORS

PART II: OFF-DIAGONAL ESTIMATES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. This is the second part of a series of four articles on weighted norm inequalities, off-diagonal estimates and elliptic operators. We consider a substitute to the notion of pointwise bounds for kernels of operators which usually is a measure of decay. This substitute is that of off-diagonal estimates expressed in terms of local and scale invariant $L^p - L^q$ estimates. We propose a definition in spaces of homogeneous type that is stable under composition. It is particularly well suited to semigroups. We study the case of semigroups generated by elliptic operators.

Contents

Date: March 15, 2006.

2000 *Mathematics Subject Classification.* 47A06 (35J15, 42B20).

Key words and phrases. Spaces of homogeneous type, off-diagonal estimates, Muckenhoupt weights, semigroups, elliptic operators.

This work was partially supported by the European Union (IHP Network "Harmonic Analysis and Related Problems" 2002-2006, Contract HPRN-CT-2001-00273-HARP). The second author was also supported by MEC "Programa Ramón y Cajal, 2005" and by MEC Grant MTM2004-00678.

1. Introduction

Since its discovery by Gaffney [Gaf] for the heat equation on Riemannian manifolds, one can show that most semigroups generated by elliptic operators satisfy the so-called L^2 off-diagonal estimates. They are of the form

$$\|T_t(\chi_E\, f)\|_{L^2(F)} \le C e^{-\frac{c\, d^2(E,F)}{t}} \|f\|_{L^2(E)}, \tag{1.1}$$

valid for all $t>0$ and all $f \in L^2$ whenever E, F are closed sets and $d(E,F)$ an appropriate distance on sets. This estimate is relevant when t is smaller than a time t_0 comparable to $d^2(E,F)=d^2$: before t_0, the heat, which we imagine with a Gaussian distribution, has not had enough time to diffuse from E to give a significant contribution on F, hence the decay in t/d^2 which explains the terminology "off-diagonal." These estimates are instrumental in many applications of semigroups. For example, they are the main technical tool (for the resolvent instead of the semigroup) in the proof of the Kato conjecture [AHLMcT]. Whether one can improve integrability properties depends on other arguments. Such an improvement, called hypercontractivity (see, e.g. [Da3]), is usually linked to some kind of Sobolev embedding.

A strong form of off-diagonal estimates is the Gaussian upper bound, that is, a pointwise control of the kernel of T_t by Gaussians:

$$|K_t(x,y)| \le C t^{-n/2} e^{-\frac{c\, d^2(x,y)}{t}}. \tag{1.2}$$

This behavior appears frequently and has yielded in the 1990's a number of beautiful results on independence of sectors of analyticity for semigroups, independence of L^p-spectrum as p varies for their generators, maximal regularity problems, ... An account on all this as well as a documented bibliographical list can be found in a recent survey by Arendt [Are].

The power of t in front of the Gaussian factor appears in homogeneous situations where the volume of balls is comparable to a power of their radii. In this case, such an estimate implies L^1-L^∞ boundedness of T_t known as the ultracontractivity property. There are geometric situations, such as Riemannian manifolds or weighted measures, where the volume of a ball is not comparable to a power of its radius. In this case, the Gaussian upper bound becomes

$$|K_t(x,y)| \le \frac{C}{\sqrt{\mathrm{Vol}\,(B(x,\sqrt{t}))\,\mathrm{Vol}\,(B(y,\sqrt{t}))}}\, e^{-\frac{c\, d^2(x,y)}{t}}, \tag{1.3}$$

and this no longer implies ultracontractivity. This estimate has to be treated as some sort of local and scale invariant L^1-L^∞ bound which can be called L^1-L^∞ off-diagonal estimates; it is still the improvement of regularity in the scale of Lebesgue spaces that matters, even if it is local.

The Gaussian upper bound (1.2) is equivalent to

$$\|T_t(\chi_E\, f)\|_{L^\infty(F)} \le C t^{-n/2} e^{-\frac{c\, d^2(E,F)}{t}} \|f\|_{L^1(E)} \tag{1.4}$$

for all $t>0$, $f \in L^1$ and all closed sets E, F. Interpolation between (1.1) and (1.4) yield intermediate $L^p - L^{p'}$ conditions of the same type. Hence, one can formulate a definition for arbitrary p, q with $1 \le p < q \le \infty$ which we call here $L^p - L^q$ full off-diagonal estimates. Such conditions appear naturally in absence of Gaussian upper

bounds (1.2). Davies showed that such a generalization already leads to improved results on independence of sectors of analyticity and independence of L^p-spectrum as p varies [Da4].

In another direction, we owe to Blunck and Kunstmann the fundamental observation that this notion of $L^p - L^q$ off-diagonal estimates for $p < q$ permits to develop a theory of singular "non-integral" operators for which one can formulate L^p boundedness criteria for p in arbitrary intervals in the absence of information on the kernels (pointwise bounds and even bounds in mean). Let us mention their weak type (p, p) criterion for $1 < p < 2$ in absence of kernels and assuming weak type $(2, 2)$, similar to the generalization by Duong and McIntosh [DMc] of Hörmander's result [Hör] when $p = 1$ in presence of kernel bounds. See their series of papers [BK1, BK2, BK3]. See also [HM] for related ideas. One can find in Fefferman's work [Fef] the essence of such a criterion but no explicit statement was needed because it was in a situation where one can split operators in pieces with localized smooth kernels. In the same spirit, [ACDH] proposes a strong type (p, p) criterion for one $p > 2$ (not all) via good-λ inequalities (we also refer the reader to the first article of the series [AM1] where we generalize this to weighted norm estimates). In [Aus], all these ideas are presented in the Euclidean setting and applied to some singular "non-integral" operators arising from elliptic operators. This yields optimal ranges of exponents p for L^p boundedness; the weighted norm extension for this application is the purpose of [AM3]. However, we observe that the $L^p - L^q$ full off-diagonal estimates when $p < q$ used by Blunck and Kunstmann, even if they are stable under composition (which is a natural property when working with semigroups), is somewhat irrealistic in general situations for at least three reasons: they implies global $L^p - L^q$ boundedness of T_t, they do not imply $L^p - L^p$ boundedness of T_t, and they do not pass to weighted estimates. So the notion of $L^p - L^q$ off-diagonal estimates, generalizing (1.3), on a space of homogeneous type for one-parameter families of operators needs a definition.

Such a definition should only involve balls and annuli and make clear that they are two parameters involved, the radius of balls and the parameter of the family, linked by a scaling rule independently on the location of the balls. Some examples suggest possible definitions (called here strong or mild off-diagonal estimates) but they are no longer stable under composition in a general context. Hence, the price to pay for stability is a somewhat weak definition (in the sense that we can not be greedy in our demands). Nevertheless, it covers examples of the literature on semigroups. Furthermore, in spaces of homogeneous type with polynomial volume growth (that is, the measure of a ball is comparable to a power of its radius, uniformly over centers and radii) it coincides with all other definitions. This is also the case for more general volume growth conditions, such as the one for some Lie groups with a local dimension and a dimension at infinity. Eventually, it is operational for proving weighted estimates in [AM3], which was the main motivation for developing this material. Since it is of independent interest, we present it here in a separate article, which can be read independently of the other papers of our series.

In Section 2, we introduce our definition of $L^p - L^q$ off-diagonal estimates on balls for one-parameter (such as time) families of operators and state the main properties: for $p = 1, q = \infty$, it is equivalent to the Gaussian upper bound (1.3) for the associated kernels, and for arbitrary p, q, it implies L^p boundedness of each operator, is stable

under composition, and passes to the weighted case (proofs are given later in Section 6). As mentioned, we discuss in Section 3 other "expected" stronger definitions for off-diagonal estimates. The proof that all our definitions coincide in spaces with polynomial volume growth is in Section 6. We then present in Section 4 the application to semigroups: we establish a correspondence between the interval of exponents p for L^p-boundedness of T_t and the set of exponents (p,q) for which $L^p - L^q$ off-diagonal estimates on balls hold when the latter set is not empty. This correspondence remains true for sectorial analytic extension of semigroups with independence of angles. Eventually, we show how unweighted off-diagonal estimates imply weighted ones for appropriate A_∞ weights. In Section 5, we apply all this and describe unweighted and weighted off-diagonal estimates on balls of semigroups $\{e^{-t\,L}\}_{t>0}$ and their gradient $\{\sqrt{t}\,\nabla\, e^{-t\,L}\}_{t>0}$ for a class of elliptic operators L in $\mathbb{R}^n$.

2. Off-diagonal estimates on balls

2.1. **Setting and notation.** Let $(\mathcal{X},d,\mu)$ be a space of homogeneous type, which is a (non empty) set $\mathcal{X}$ endowed with a distance d (it could even be a quasi-distance but we restrict to this situation for simplicity) and a non-negative Borel measure μ on $\mathcal{X}$ such that the doubling condition

$$\mu(B(x,2\,r)) \le C_0\,\mu(B(x,r)) < \infty, \tag{2.1}$$

holds for all $x \in \mathcal{X}$ and $r>0$, where $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$.

Throughout this paper we use the following notation: for every ball B, x_B and r_B are respectively its center and its radius, that is, $B = B(x_B, r_B)$. Given $\lambda > 0$, we will write $\lambda\,B$ for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda\, r_B$.

If C_0 is the smallest constant for which the measure μ verifies the doubling condition (2.1), then $D = \log_2 C_0$ is called the doubling order of μ and we have that $\mu(\lambda\,B) \le C_\mu\,\lambda^D\,\mu(B)$, for every ball B and for every $\lambda \ge 1$.

Given a ball B we set $C_j(B) = 2^{j+1}\,B \setminus 2^j\,B$ for $j \ge 2$; $C_1(B) = 4\,B$ and also $\widehat{C}_1(B) = 4\,B \setminus 2\,B$. We set

$$\fint_B h\,d\mu = \frac{1}{\mu(B)}\int_B h\,d\mu, \qquad\qquad \fint_{B^c} h\,d\mu = \frac{1}{\mu(2\,B)}\int_{B^c} h\,d\mu,$$

and for $j \ge 1$

$$\fint_{C_j(B)} h\,d\mu = \frac{1}{\mu(2^{j+1}\,B)}\int_{C_j(B)} h\,d\mu.$$

The last notation can be seen as the average on $2^{j+1}B$ of $\chi_{C_j(B)}\,h$, where we denote by χ_E the indicator function of a set E. It is not necessarily the case that $2^{j+1}\,B$ and $C_j(B)$ have comparable masses with constant independent of B and j (for example, when $C_j(B) = \emptyset$) so it is safer to divide out by the mass of the larger set (which is never 0 unless $\mu = 0$) and fortunately, this is the quantity arising in computations. Although this is not needed in this work, let us mention a reasonable sufficient condition insuring this comparability (see [AM1] for a proof): Assume that there exists $\varepsilon \in (0,1)$ such that for any ball $B \subset \mathcal{X}$, $(2-\varepsilon)\,B \setminus B \ne \emptyset$. Then, $\mu(2\,B) \lesssim \mu(2\,B \setminus B)$ for any ball B, where the implicit constants are independent of B.

For shortness we write $\Upsilon(s)=\max\{s,s^{-1}\}$ for $s>0$. We use the symbol $A\lesssim B$ for $A\le CB$ for some constant C whose value is not important and independent of the parameters at stake.

2.2. **Definition and comments.**

Definition 2.1. *Given $1\le p\le q\le\infty$, we say that a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p(\mu)-L^q(\mu)$ off-diagonal estimates on balls, which by an abuse of notation will be denoted $T_t\in\mathcal{O}\big(L^p(\mu)-L^q(\mu)\big)$, if there exist $\theta_1,\theta_2>0$ and $c>0$ such that for every $t>0$ and for any ball B, setting $r=r_B$,*

$$\Big(\fint_B |T_t(\chi_B\, f)|^q\,d\mu\Big)^{\frac1q}\lesssim\Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\theta_2}\,\Big(\fint_B |f|^p\,d\mu\Big)^{\frac1p};\tag{2.2}$$

and, for all $j\ge2$,

$$\Big(\fint_B |T_t(\chi_{C_j(B)}\, f)|^q\,d\mu\Big)^{\frac1q}\lesssim 2^{j\,\theta_1}\Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\theta_2}\,e^{-\frac{c\,4^j\,r^2}{t}}\Big(\fint_{C_j(B)} |f|^p\,d\mu\Big)^{\frac1p}\tag{2.3}$$

and

$$\Big(\fint_{C_j(B)} |T_t(\chi_B\, f)|^q\,d\mu\Big)^{\frac1q}\lesssim 2^{j\,\theta_1}\Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\theta_2}\,e^{-\frac{c\,4^j\,r^2}{t}}\Big(\fint_B |f|^p\,d\mu\Big)^{\frac1p}.\tag{2.4}$$

Comments.

1. When $q=\infty$ one has to change the L^q-norms by the corresponding essential suprema.
2. T_t may only be defined on a subspace of $L^p(\mu)$ provided this subspace is stable under truncation by indicator functions of measurable sets (balls would suffice for the definition but measurable sets is needed for interpolation). In this case, it is understood that the definition applies to functions f in this subspace.
3. Even though our definition makes sense when $p\ge q\ge1$, we restrict ourselves to $p\le q$ to stress the regularizing effect in the scale of Lebesgue spaces.
4. Hölder's inequality implies $\mathcal{O}\big(L^p(\mu)-L^q(\mu)\big)\subset\mathcal{O}\big(L^{p_1}(\mu)-L^{q_1}(\mu)\big)$ for all p_1,q_1 with $p\le p_1\le q_1\le q$.
5. $T_t\in\mathcal{O}\big(L^p(\mu)-L^q(\mu)\big)$ with $p<q$ does not imply that T_t is bounded from $L^p(\mu)$ into $L^q(\mu)$.
6. If T_t is linear and defined on a dense subspace of $L^p(\mu)$, then $T_t\in\mathcal{O}\big(L^p(\mu)-L^q(\mu)\big)$ if and only if $T_t^*\in\mathcal{O}\big(L^{q'}(\mu)-L^{p'}(\mu)\big)$ where T_t^* is the dual operator for the duality form $\int_{\mathcal{X}} fg\,d\mu$.
7. Given two Banach spaces B_1 and B_2, this definition, with the corresponding changes, is also valid for operators taking B_1-valued functions into B_2-valued functions.
8. The chosen "time-space" scaling $\frac{r}{\sqrt{t}}$ is irrelevant. It can be changed at will to $\frac{r}{t^\gamma}$ for any $\gamma>0$ simply by changing T_t to $T_{t^{2\gamma}}$. This scaling corresponds to semigroups of second order operators, which is our main application in [AM3].

9. The profile $s \mapsto e^{-c\,s^2}$ can be replaced by any non increasing $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\theta \geq 0$, we have $s^\theta\, g(s) \to 0$ as $s \to \infty$. For example $g(s) = e^{-c\,s^\alpha}$ with $c, \alpha > 0$ is acceptable. The value of c has no interest to us provided it remains non negative. Thus, we will freely use the same c from line to line. Profiles with sufficiently large polynomial decay $(1+s)^{-p}$ works as well, but p would have to be adjusted to each application.

10. For $s \leq 1$, $\Upsilon(s)^{\theta_2}\, e^{-cs^2}$ is comparable to $s^{-\theta_2}$. Since s is to be replaced by $\frac{2^j\, r}{\sqrt{t}}$, it is curious at first sight that we allow such negative powers; imposing positive powers for small s seems more natural. We were forced into negative powers to obtain stability under composition. See Lemma 6.3 below for the technical reason. Fortunately, this apparently weak behavior is sufficient for our applications in [AM3].

11. One can replace θ_1 by $\theta_1 + \alpha$ for any $\alpha \geq 0$ and the same happens with θ_2. In fact, making $\theta_2 \geq \theta_1$, one obtains an equivalent definition by replacing $2^{j\,\theta_1}\Upsilon\Big(\frac{2^j\, r}{\sqrt{t}}\Big)^{\theta_2}\, e^{-\frac{c\,4^j\, r^2}{t}}$ by an expression of the form $\Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\theta_2}\, e^{-\frac{c\,4^j\, r^2}{t}}$ (up to changing the c's). We stick to the first formulation for simplicity in some calculations but this is a first indication that the value of the exponent θ_1 is irrelevant.

12. Definition 2.1 is given in terms of dyadic annuli but an equivalent definition can be written in terms of a-adic annuli for all $a > 1$. See the proof of Lemma 6.5 for a possible argument.

2.3. **The case $p = 1$ and $q = \infty$.** We first state that the definition for $p = 1$ and $q = \infty$ coincides with the usual pointwise Gaussian decay of the introduction.

Proposition 2.2. *Assume that the operators T_t, $t > 0$, are linear. Then $T_t \in \mathcal{O}\big(L^1(\mu) - L^\infty(\mu)\big)$ if and only if there exist constants $C, c > 0$ and for each $t > 0$, a measurable function K_t on $\mathcal{X} \times \mathcal{X}$ such that $T_t f(x) = \int_{\mathcal{X}} K_t(x,y) f(y)\, d\mu(y)$ holds for almost every $x \in \mathcal{X}$ whenever $f \in L^1(\mu)$ and for almost every $(x,y) \in \mathcal{X} \times \mathcal{X}$,*

$$|K_t(x,y)| \leq \frac{C}{\mu(B(x,\sqrt{t}))}\, e^{-\frac{c\, d^2(x,y)}{t}}. \tag{2.5}$$

The (easy) proof in Section 6.1 shows the role of the independence of r and t in the condition $T_t \in \mathcal{O}\big(L^1(\mu) - L^\infty(\mu)\big)$ and of the scaling rule $r/\sqrt{t}$. It also shows the irrelevance of the exponents θ_1, θ_2 in this case. Note that the doubling condition on μ implies that the Gaussian expressions in (1.3) and (2.5) are comparable up to changing the constants $C, c > 0$.

2.4. **Uniform boundedness and stability under composition.** We state here the most important features of this notion: Off-diagonal estimates on balls imply uniform boundedness and are stable under composition.

Theorem 2.3.

(a) *If $T_t \in \mathcal{O}\big(L^p(\mu) - L^p(\mu)\big)$ then $T_t : L^p(\mu) \longrightarrow L^p(\mu)$ is bounded uniformly on t.*

(*b*) *Let* $1 \le p \le q \le r \le \infty$. *If* $T_t \in \mathcal{O}\big(L^q(\mu) - L^r(\mu)\big)$ *and* $S_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ *then* $T_t \circ S_t \in \mathcal{O}\big(L^p(\mu) - L^r(\mu)\big)$. *Furthermore if* θ_1, θ_2 *are the exponents appearing in Definition 2.1 for* T_t *and* γ_1, γ_2 *are the ones for* S_t, *then the composition* $T_t \circ S_t \in \mathcal{O}\big(L^p(\mu) - L^r(\mu)\big)$ *satisfies the corresponding inequalities with* $\alpha_1 = \max\{\theta_1, \gamma_1, D/r\}$ *and* $\alpha_2 = \max\{\theta_1, \theta_2\} + \max\{\gamma_1 + D/q, \gamma_2\}$. †

In (*b*), if T_t and S_t are defined on subspaces, this result is understood in the sense that one restricts to functions f for which $T_t \circ S_t f$ is well-defined.

The proof of this result can be found in Sections 6.2 and 6.3.

2.5. **Weighted off-diagonal estimates.** Weighted off-diagonal estimates on balls with weights in the Muckenhoupt class A_∞ can be obtained from the off-diagonal estimates on balls with respect to the underlying measure.

We use the following notation: given a weight w we consider the measure $dw = w\, d\mu$, so $w(E) = \int_E dw = \int_E w\, d\mu$ and also $L^p(w) = L^p(w\, d\mu)$. Notice that the notation for the averages used before depends on the measure we are using and so

$$\fint_B h\, d\mu = \frac{1}{\mu(B)} \int_B h\, d\mu, \qquad\qquad \fint_B h\, dw = \frac{1}{w(B)} \int_B h\, dw = \frac{1}{w(B)} \int_B h\, w\, d\mu,$$

the same happens for "averages" on B^c and $C_j(B)$ as defined in Section 2.1.

Let $w \in A_\infty$ (we recall some basic facts about A_p and RH_s weights in Appendix A). Since $(\mathcal{X}, d, \mu)$ is a space of homogeneous type, the measure w is doubling and $(\mathcal{X}, d, w)$ is also a space of homogeneous type. Hence, off-diagonal estimates make sense in that space.

Proposition 2.4. *Let* $1 \le p_0 < q_0 \le \infty$ *and* $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ *for all* p, q *with* $p_0 < p \le q < q_0$. *Then, for all* p, q *with* $p_0 < p \le q < q_0$ *and for any* $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{q})'}$ *we have that* $T_t \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$.

The proof of this result can be found in Section 6.4.

3. Other types of off-diagonal estimates

3.1. **Full off-diagonal estimates.** In the case where $(\mathcal{X}, d, \mu)$ is the usual Euclidean space with Lebesgue measure or more generally, a group with polynomial volume growth (we say that $(\mathcal{X}, d, \mu)$ has polynomial volume growth when $\mu(B(x,r)) \sim r^n$ for some $n > 0$ and uniformly for all $x \in \mathcal{X}$ and $r > 0$), one encounters more precise off-diagonal estimates. This yields a possible definition in spaces of homogeneous type.

Definition 3.1. *Let* $(\mathcal{X}, d, \mu)$ *be a space of homogeneous type. Let* $1 \le p \le q \le \infty$. *We say that a family* $\{T_t\}_{t>0}$ *of sublinear operators satisfies* $L^p(\mu) - L^q(\mu)$ ***full off-diagonal estimates****, in short* $T_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$, *if for some constant* $\theta \ge 0$, *with* $\theta \ne 0$ *when* $p < q$, *for all closed sets* E *and* F, *all* f *and all* $t > 0$ *we have*

$$\Big(\int_F |T_t(\chi_E\, f)|^q\, d\mu\Big)^{\frac{1}{q}} \lesssim t^{-\theta} e^{-\frac{c d^2(E,F)}{t}} \Big(\int_E |f|^p\, d\mu\Big)^{\frac{1}{p}}. \tag{3.1}$$

†When $\theta_1 \ne \theta_2$ and $\gamma_1 + D/q \ne \gamma_2$, the value of α_2 is correct. Otherwise, α_2 is any number strictly bigger than this value, see Remark 6.4 below.

Again, the operators are defined on some subspace $\mathcal{D}$ that is stable under truncation by indicators of measurable sets. Full off-diagonal estimates appear when dealing with semigroups of second order elliptic operators (see [Gaf, Da2, LSV, Aus] ...). The most studied case is when $p = 1$ and $q = \infty$ which means that the kernel of T_t has pointwise Gaussian upper bounds (see [Aro, FS, Cou, VSC, Da3, Rob, AMcT, AT, AE, DER]...). If one considers higher order operators, then t changes to some positive power of t and the Gaussian to other exponential like function (if t denotes time) (see [Da4, AT]). Our t here may not be the usual time scale and the Gaussian may be changed also. We stick to this case to keep the presentation simple.

When $\mathcal{X} = \mathbb{R}^n$, the usual value of θ (given our choice of "space-time" scaling) is $\theta = \frac{1}{2}\big(\frac{n}{p} - \frac{n}{q}\big)$. See the proof of Proposition 3.2.

Here is a list of simple and known facts whose proofs will be left to the reader.

(a) $T_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$ implies T_t bounded from $L^p(\mu)$ to $L^q(\mu)$.

(b) If $p \le r \le q$, $S_t \in \mathcal{F}\big(L^p(\mu) - L^r(\mu)\big)$ and $T_t \in \mathcal{F}\big(L^r(\mu) - L^q(\mu)\big)$, then $T_t \circ S_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$.

Let us compare this definition with the previous one. On the one hand, if $q = p$ then $\mathcal{F}\big(L^p(\mu) - L^p(\mu)\big)$ easily implies $\mathcal{O}\big(L^p(\mu) - L^p(\mu)\big)$. The converse is true and follows from the proof of Proposition 3.2, (b), stated below. No further condition on the space is needed at this point.

On the other hand, these notions cease to be comparable when $p < q$ without further information on the space $\mathcal{X}$. Assume that $T_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$. If $E = F = B$ is a ball, then

$$\Big(\fint_B |T_t(\chi_B\, f)|^q\, d\mu\Big)^{\frac{1}{q}} \lesssim t^{-\theta}\, \mu(B)^{\frac{1}{p}-\frac{1}{q}} \Big(\fint_B |f|^p\, d\mu\Big)^{\frac{1}{p}}.$$

Unless there is some control from above of $\mu(B)$ by a power of the radius of B, we cannot conclude that $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$. Similarly unless there is such a control, one cannot conclude that T_t uniformly bounded on $L^r(\mu)$ for $p \le r \le q$. Eventually, if $p \le p_1 < q_1 \le q$, we do not know if $T_t \in \mathcal{F}\big(L^{p_1}(\mu) - L^{q_1}(\mu)\big)$.

As $L^p(\mu) - L^q(\mu)$ full off-diagonal estimates when $p < q$ imply $L^p(\mu) - L^q(\mu)$ boundedness but not $L^p(\mu)$ boundedness, this is not an encountered notion on a general space of homogenous type. For example, the heat semigroup $e^{-t\Delta}$ on functions for general Riemannian manifolds with the doubling property is not $L^p - L^q$ bounded when $p < q$ unless, as Proposition 3.2 will show, the measure of any ball is bounded below by a power of its radius.

Here is a statement that connects both notions. The proof is postponed to Section 6.5.

Proposition 3.2. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $1 \le p < q \le \infty$.*

(a) *Assume that $\mathcal{X}$ has volume growth at most polynomial, that is, $\mu(B(x,r)) \lesssim r^n$ for some $n > 0$ and uniformly for all $x \in \mathcal{X}$ and $r > 0$. If $T_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$ with exponent θ in* (3.1) *equal to $\frac{1}{2}\big(\frac{n}{p} - \frac{n}{q}\big)$ then $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$.*

(b) *Assume that $\mathcal{X}$ has volume growth at least polynomial, that is, $\mu(B(x,r)) \gtrsim r^n$ for some $n > 0$ and uniformly for all $x \in \mathcal{X}$ and $r > 0$. If $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ then $T_t \in \mathcal{F}\big(L^p(\mu) - L^q(\mu)\big)$ with exponent θ in* (3.1) *equal to $\frac{1}{2}\big(\frac{n}{p} - \frac{n}{q}\big)$.*

Let us go a little further. We say that a space of homogeneous type is of φ-growth if $\mu(B(x,r)) \sim \varphi(r)$ uniformly for $x \in \mathcal{X}$ and $r > 0$, where φ is a non-decreasing function on $(0,\infty)$. Remark that the fact that space is of homogeneous type implies that φ is doubling in the sense that $\sup_{r>0} \frac{\varphi(2r)}{\varphi(r)} < \infty$.[†] A particular important example is the Heisenberg group equipped with Riemannian distance and Haar measure: in this case, $\varphi(r) \sim r^d$ for $r \le 1$ and $\varphi(r) \sim r^D$ for $r \ge 1$, the exponents $d > 0$ and $D > 0$ being called respectively its local dimension and its dimension at infinity. Call "$L^p(\mu) - L^q(\mu)$ full off-diagonal estimates of type φ" the estimates of Definition 3.1 with $t^{-\theta}$ replaced by $\varphi\big(\sqrt{t}\,\big)^{\frac{1}{q}-\frac{1}{p}}$ for all $t > 0$.

Proposition 3.3. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type of φ-growth and $1 \le p < q \le \infty$. Then $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ if and only if T_t satisfies $L^p(\mu) - L^q(\mu)$ full off-diagonal estimates of type φ.*

In other words, the scaling $\frac{r}{\sqrt{t}}$ contained in the off-diagonal estimates on balls plus the volume growth completely rule the function of $\sqrt{t}$ in the full off-diagonal estimates.

The proof of this result is postponed to Section 6.5.

Our last remark is that full off-diagonal estimates do not pass to weighted measures as well: for example, in $\mathbb{R}^n$, the power weights $w(x) = |x|^{-\alpha}$ for $0 < \alpha < n$ are neither with polynomial growth from below or above nor with φ-growth.

3.2. **Mild off-diagonal estimates on balls.** As in Section 2.5 full off-diagonal estimates imply some (but not full) off-diagonal weighted estimates for an appropriate class of weights. Assume that $\mathcal{X} = \mathbb{R}^n$ equipped with Lebesgue measure. With the same arguments (see Section 6.4) we obtain that if $1 \le p_0 < q_0 \le \infty$ and T_t satisfies $L^p(dx) - L^q(dx)$ full off-diagonal estimates (3.1) for all p, q with $p_0 < p \le q < q_0$ and $\theta = \frac{1}{2}\big(\frac{n}{p} - \frac{n}{q}\big)$, then, for all p, q with $p_0 < p \le q < q_0$ and for any $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{q})'}$ we have that

$$\Big(\fint_{B'} \chi_{K'}\, |T_t f|^q\, dw\Big)^{\frac{1}{q}} \lesssim c(t, B, B', K, K') \Big(\fint_B \chi_K\, |f|^p\, dw\Big)^{\frac{1}{p}} \tag{3.2}$$

where

$$c(t, B, B', K, K') = \left(\frac{\sqrt{t}}{r_{B'}}\right)^{\frac{n}{q_1}} \left(\frac{r_B}{\sqrt{t}}\right)^{\frac{n}{p_1}} e^{-\frac{c\, d^2(K', K)}{t}}$$

whenever B, B' are balls, K, K' are respective compact subsets, f bounded with support in K, $t > 0$ and p_1, q_1 are some numbers chosen with $p_0 < p_1 < q_1 < q_0$. For $q = \infty$, the left hand side of (3.2) is understood as the essential supremum on B'. If we specialize to the three cases of Definition 2.1, namely, 1) $B = B' = K = K'$, 2) $B = K, B' = 2^{j+1}B, K' = C_j(B)$ and 3) the symmetric case of 2), we obtain (2.2), (2.3), (2.4) with $\theta_1 = n/q_1$ and $\theta_2 = n/p_1 - n/q_1 > 0$ and $\Upsilon(s)$ is replaced by s.

This leads us to another definition of off-diagonal estimates in a general context.

Definition 3.4. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. Let $1 \le p \le q \le \infty$. We say that a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p(\mu) - L^q(\mu)$* ***mild off-diagonal estimates on balls*** *if there exist real numbers $\theta_1, \theta_2 \ge 0$, $c > 0$ with $\theta_2 > 0$ when $p < q$ such that* (2.2), (2.3) *and* (2.4) *hold with s replacing $\Upsilon(s)$.*

[†]We think that the discussion can be extended somehow to spaces with exponential growth, but this is beyond the scope of the present article.

Remark 3.5. In replacing $\Upsilon(s)$ by s then one cannot enlarge θ_2 at will as in the definition of off-diagonal estimates on balls. Hence, the restriction that θ_2 should be non negative when $p<q$ seems meaningful.

This is clearly stronger than Definition 2.1 since we impose the power of s to be positive even for small s (see comment 10 after Definition 2.1). However, stability under composition is unclear. If S_t satisfies $L^p(\mu)-L^q(\mu)$ mild off-diagonal estimates on balls and T_t satisfies $L^q(\mu)-L^r(\mu)$ mild off-diagonal estimates on balls, then we do not know whether $T_t\circ S_t$ satisfies $L^p(\mu)-L^r(\mu)$ mild off-diagonal estimates on balls. Of course, $T_t\circ S_t\in\mathcal{O}\big(L^p(\mu)-L^r(\mu)\big)$ (hence, under φ-growth there is stability)

We may have lost too much information in passing from full off-diagonal estimates to mild off-diagonal estimates on balls, hence the lack of stability. In particular, we restricted attention to balls while the closed sets E and F in (3.1) could be unbounded.

3.3. **Strong off-diagonal estimates on balls.** The following result will suggest an even stronger definition.

Proposition 3.6. *Assume that $(\mathcal{X},d,\mu)$ is the usual Euclidean space $\mathbb{R}^n$ with Lebesgue measure. Fix $1\le p_0<q_0\le\infty$. Assume that $\{T_t\}_{t>0}$ satisfies $L^p(dx)-L^q(dx)$ full off-diagonal estimates for all p,q with $p_0<p\le q<q_0$ and $\theta=\frac{1}{2}\big(\frac{n}{p}-\frac{n}{q}\big)$. Fix p,q with $p_0<p\le q<q_0$ and assume that $w\in A_{\frac{p}{p_0}}\cap RH_{(\frac{q_0}{q})'}$. Let B be a ball and set $r=r_B$. Then for all f,*

$$\Big(\fint_{(2B)^c}|T_t(\chi_B\,f)|^q\,dw\Big)^{\frac{1}{q}}\lesssim\Big(\frac{r}{\sqrt{t}}\Big)^{\beta}e^{-\frac{c\,r^2}{t}}\Big(\fint_B|f|^p\,dw\Big)^{\frac{1}{p}}\tag{3.3}$$

and

$$\Big(\fint_B|T_t(\chi_{(2B)^c}\,f)|^q\,dw\Big)^{\frac{1}{q}}\lesssim\Big(\frac{r}{\sqrt{t}}\Big)^{\gamma}e^{-\frac{c\,r^2}{t}}\Big(\fint_{(2B)^c}|f|^p\,dw\Big)^{\frac{1}{p}}\tag{3.4}$$

with $\beta,\gamma\ge0$ and non zero when $p<q$.

The proof of this result is postponed until Section 6.6.

Definition 3.7. *Let $(\mathcal{X},d,\mu)$ be a space of homogeneous type. Let $1\le p\le q\le\infty$. We say that a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p(\mu)-L^q(\mu)$* ***strong off-diagonal estimates on balls*** *if there exist real numbers $\alpha\ge0$ with $\alpha>0$ when $p<q$ and $c>0$ such that for any ball B and any $t>0$, setting $r=r_B$, and any f in an appropriate space $\mathcal{D}$,*

$$\Big(\fint_B|T_t(\chi_B\,f)|^q\,d\mu\Big)^{\frac{1}{q}}\lesssim\Big(\frac{r}{\sqrt{t}}\Big)^{\alpha}\Big(\fint_B|f|^p\,d\mu\Big)^{\frac{1}{p}};\tag{3.5}$$

$$\Big(\fint_{(2B)^c}|T_t(\chi_B\,f)|^q\,d\mu\Big)^{\frac{1}{q}}\lesssim\Big(\frac{r}{\sqrt{t}}\Big)^{\alpha}e^{-\frac{c\,r^2}{t}}\Big(\fint_B|f|^p\,d\mu\Big)^{\frac{1}{p}};\tag{3.6}$$

$$\Big(\fint_B|T_t(\chi_{(2B)^c}\,f)|^q\,d\mu\Big)^{\frac{1}{q}}\lesssim\Big(\frac{r}{\sqrt{t}}\Big)^{\alpha}e^{-\frac{c\,r^2}{t}}\Big(\fint_{(2B)^c}|f|^p\,d\mu\Big)^{\frac{1}{p}}.\tag{3.7}$$

It is clear that strong off-diagonal estimates on balls imply mild off-diagonal estimates on balls: for instance, to get the analog of (2.3), we write $\widetilde{B}=2^{j-1}B$ and note that $C_j(B)\subset(2\,\widetilde{B})^c$. So we apply (3.7) with $\widetilde{B}$ and then we obtain (2.3) with s replacing $\Upsilon(s)$, $\theta_2=\alpha$ and $\theta_1=D/q$. The same can be done in the other cases.

It is also interesting to compare the last two inequalities of this definition with the ones in Lemma 6.6: again $\Upsilon(s)$ is replaced by s. We also stress that such a definition implies the partial $L^p(\mu) - L^q(\mu)$ boundedness inequalities

$$\Big(\fint_B |T_t f|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim \Big(\frac{r}{\sqrt{t}}\Big)^\alpha \Big(\frac{1}{\mu(B)} \int_{\mathcal{X}} |f|^p \, d\mu\Big)^{\frac{1}{p}}$$

and

$$\Big(\frac{1}{\mu(B)} \int_{\mathcal{X}} |T_t(\chi_B \, f)|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim \Big(\frac{r}{\sqrt{t}}\Big)^\alpha \Big(\fint_B |f|^p \, d\mu\Big)^{\frac{1}{p}}.$$

Had we put an estimate from B^c to B^c similar to (3.5) then we would derive global $L^p(\mu) - L^q(\mu)$ boundedness, which is not realistic when $p < q$ in a non polynomial growth situation.

For this reason precisely, strong off-diagonal estimates do not compose well. The best we can say (even if we allow the exponent α to take different values in (3.5), (3.6), (3.7)) is: If S_t satisfies $L^p(\mu) - L^q(\mu)$ strong off-diagonal estimates on balls and T_t satisfies $L^q(\mu) - L^r(\mu)$ strong off-diagonal estimates on balls then $T_t \circ S_t$ satisfies $L^p(\mu) - L^r(\mu)$ mild off-diagonal estimates on balls (this can be obtained easily following the proof of (b) in Theorem 2.3 and using the definition of the strong off-diagonal estimates on balls in place of Lemma 6.6). Again, assuming φ-growth, there is stability under composition, passing via full off-diagonal estimates.

In conclusion, using only balls, complements of balls and annuli for defining off-diagonal estimates (instead of closed sets) forces us into an apparently weak definition to have stability under composition. But under a polynomial (or φ-) growth, all these notions are the same.

4. Propagation and semigroups

We are interested in values of p, q for which $L^p - L^q$ off-diagonal estimates on balls hold, especially when there is a regularizing effect, that is, when $p < q$.

4.1. **Propagation property.** Let $\mathcal{T} = \{T_t\}_{t>0}$ be a family of sublinear operators defined on a space $\mathcal{D}$ contained in all $L^p(\mu)$ that is stable under truncation by indicator functions of measurable sets.

Let $\widetilde{\mathcal{J}}(\mathcal{T})$ be the interval of all exponents $p \in [1, \infty]$ such that T_t is bounded uniformly with respect t on $L^p(\mu)$.

We introduce the set

$$\mathcal{O}(\mathcal{T}) = \{(p, q) \in [1, \infty]^2 \, ; \, p < q, \; T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)\}.$$

If we set $\mathcal{C}(\mathcal{T}) = \{(\frac{1}{p}, \frac{1}{q}) \, ; \, (p, q) \in \mathcal{O}(\mathcal{T})\}$, then by interpolation, it is a convex set contained in $\{(u, v) \in [0, 1]^2 \, ; \, u > v\}$.

The relation between $\mathcal{O}(\mathcal{T})$ and $\widetilde{\mathcal{J}}(\mathcal{T})$ is the following. If $(p, q) \in \mathcal{O}(\mathcal{T})$ then the interval $[p, q]$ is contained in $\widetilde{\mathcal{J}}(\mathcal{T})$. This fact is a consequence of Theorem 2.3, part (a).

Also, if $\mathcal{O}(\mathcal{T}) \neq \emptyset$, then for $p \in \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$,[†] there exists $q = q(p) > p$ such that $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$. In other words, $L^p(\mu)$ boundedness improves into some

[†]If E is a subset of $[1, \infty]$ with lower and upper bound p, q then we set $\operatorname{Int} E = (p, q) = \{t \in \mathbb{R} \, ; \, p < t < q\}$, which is the interior of $E \cap \mathbb{R}$ in $\mathbb{R}$.

off-diagonal estimates on balls with increase of exponent or, differently, off-diagonal estimates on balls for one pair (p, q) propagate to pairs $(p, q(p))$ for all $p \in \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$.[‡] Indeed, let $p \in \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$. Let $(q, r) \in \mathcal{O}(\mathcal{T})$. If $p = q$ we have finished. Otherwise, we have that $p, q \in \widetilde{\mathcal{J}}(\mathcal{T})$, and since p is in the interior, there exists $\tilde{p} \in \widetilde{\mathcal{J}}(\mathcal{T})$ such that p lies in the open interval between $\tilde{p}$ and q. The $L^{\tilde{p}}(\mu)$ boundedness implies that T_t satisfies (2.2), (2.3) and (2.4) with $q = p = \tilde{p}$, $\theta_1 = D/\tilde{p}$, $\theta_2 = 0$ and $c = 0$. We interpolate (by the real method since we allow sublinear operators) these estimates with the ones coming from $L^q(\mu) - L^r(\mu)$ off-diagonal estimates on balls. Thus $(q_\theta, r_\theta) \in \mathcal{O}(\mathcal{T})$ where $1/q_\theta = \theta/\tilde{p} + (1-\theta)/q$, $1/r_\theta = \theta/\tilde{p} + (1-\theta)/r$ and $\theta \in (0, 1)$. Choosing θ such that $q_\theta = p$ proves that $(p, q(p)) \in \mathcal{O}(\mathcal{T})$ with $q(p) = r_\theta > p$.

In general, $\mathcal{C}(\mathcal{T})$ has no further structure. For example on $\mathbb{R}^n$ equipped with Lebesgue measure, let T_t be the operator of convolution with $t^{-n/2}\phi(x/\sqrt{t})$ with $t > 0$, ϕ positive, supported in the unit ball and $\phi \in L^s$ if and only if $1 \le s \le \rho$ for some $\rho \in (1, \infty)$. From Young's inequality, it is easy to determine $\mathcal{C}(\mathcal{T})$ as the region in $[0, 1]^2$ below the diagonal $v = u$ (excluded) and above the line $v = u - 1/\rho'$ (included) and also to find that $\widetilde{\mathcal{J}}(\mathcal{T}) = [1, \infty]$. In particular, there is no interval I in $[1, \infty]$ such that for all p, q with $p < q$, $p, q \in I$ is equivalent to $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$, in such a case $\mathcal{C}(\mathcal{T})$ (and $\mathcal{O}(\mathcal{T})$) would be a triangle.

4.2. **Application to semigroups.** Let $\mathbb{B}$ be a Banach space of measurable functions stable under truncations with indicator functions of measurable sets and containing all simple functions. In this way, $\mathbb{B} \cap L^p(\mu)$ is dense in $L^p(\mu)$ for all $1 \le p < \infty$.

Let $\{T_t\}_{t>0}$ be a semigroup of bounded linear operators on $\mathbb{B}$, that is, we assume for $t, s > 0$ that

$$T_t \in \mathcal{L}(\mathbb{B}); \qquad\qquad T_s \circ T_t = T_{s+t}.$$

Here and in what follows $\mathcal{L}(X)$ denotes the set of bounded linear operators on a Banach space X.

Proposition 4.1. *Set $\mathcal{T} = \{T_t\}_{t>0}$. Assume there exist $\tilde{p}, \tilde{q}$ with $1 \le \tilde{p} < \tilde{q} \le \infty$ such that $T_t \in \mathcal{O}\big(L^{\tilde{p}}(\mu) - L^{\tilde{q}}(\mu)\big)$.*[†] *Then, there exists a unique subset of $[1, \infty]$, which we denote by $\mathcal{J}(\mathcal{T})$, such that the following holds:*

$$\forall\, p, q \in [1, \infty],\ p < q \quad \Big(\ T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big) \Longleftrightarrow p, q \in \mathcal{J}(\mathcal{T})\ \Big). \tag{4.1}$$

This set is an interval, contains $[\tilde{p}, \tilde{q}]$, $\mathcal{J}(\mathcal{T}) \subset \widetilde{\mathcal{J}}(\mathcal{T})$ and $\operatorname{Int} \mathcal{J}(\mathcal{T}) = \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$.

Remark 4.2. This propagation property is reminiscent of the extrapolation for L^p–L^q boundedness developed for semigroups in [Cou].

With the notation of the previous section, (4.1) reformulates into

$$\forall\, p, q \in [1, \infty],\ p < q \quad \Big(\ (p, q) \in \mathcal{O}(\mathcal{T}) \Longleftrightarrow p, q \in \mathcal{J}(\mathcal{T})\ \Big),$$

which means that $\mathcal{O}(\mathcal{T})$ is a triangle.

Proof. Note that if E, F are two subsets such that (4.1) holds for E and F then, clearly, $E = F$ and so the uniqueness follows. Let us now construct such a set.

[‡] Here, we see p as the exponent in the source space. It could also be taken as the exponent of the target space: for $q \in \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$, there exists $p = p(q) < q$ such that $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$.

[†] It is understood that the functions to be considered are in $L^{\tilde{p}}(\mu) \cap \mathbb{B}$.

Fix $\tilde p < \tilde r < \tilde q$. Let $\mathcal{J}_-(\mathcal{T})$ be the set of all $p \in [1, \tilde r]$ such that $(p, \tilde r) \in \mathcal{O}(\mathcal{T})$. By one of the remarks after Definition 2.1, this set is an interval with upper bound $\tilde r$ and it contains $[\tilde p, \tilde r]$. Similarly, the set $\mathcal{J}_+(\mathcal{T})$ of all $p \in [\tilde r, \infty]$ such that $(\tilde r, p) \in \mathcal{O}(\mathcal{T})$, is an interval containing $[\tilde r, \tilde q]$. Set $\mathcal{J}(\mathcal{T}) = \mathcal{J}_-(\mathcal{T}) \cup \mathcal{J}_+(\mathcal{T})$. This is clearly an interval and it contains $[\tilde p, \tilde q]$.

Let us see that $p, q \in \mathcal{J}(\mathcal{T})$ with $p < q$ imply $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$. Indeed, if $p < q \le \tilde r$ or $\tilde r \le p < q$, then $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ using one of the remarks after Definition 2.1, hence $(p, q) \in \mathcal{O}(\mathcal{T})$. If $p \le \tilde r < q$, then $T_t \in \mathcal{O}\big(L^p(\mu) - L^{\tilde r}(\mu)\big)$ and $T_t \in \mathcal{O}\big(L^{\tilde r}(\mu) - L^q(\mu)\big)$ hence by the semigroup property and Theorem 2.3, part (b), $T_{2t} \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$. But we may change $2t$ to t and we have $(p, q) \in \mathcal{O}(\mathcal{T})$.

We prove the converse: let $1 \le p < q \le \infty$ with $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ and let us show that $p, q \in \mathcal{J}(\mathcal{T})$.

Case $q \le \tilde r$**:** We have $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ and $T_t \in \mathcal{O}\big(L^{\tilde r}(\mu) - L^{\tilde r}(\mu)\big)$. Hence, by interpolation, $T_t \in \mathcal{O}\big(L^{p_\theta}(\mu) - L^{q_\theta}(\mu)\big)$ where $1/p_\theta = \theta/p + (1-\theta)/\tilde r$, $1/q_\theta = \theta/q + (1-\theta)/\tilde r$ and $\theta \in (0,1)$. If $p < \inf \mathcal{J}_-(\mathcal{T})$ then we can choose θ such that $p_\theta < \inf \mathcal{J}_-(\mathcal{T}) < q_\theta$. Since $\mathcal{J}_-(\mathcal{T})$ is an interval, $q_\theta \in \mathcal{J}_-(\mathcal{T})$, that is, $T_t \in \mathcal{O}\big(L^{q_\theta}(\mu) - L^{\tilde r}(\mu)\big)$. By Theorem 2.3, part (b), and the semigroup property, $T_{2t} \in \mathcal{O}\big(L^{p_\theta}(\mu) - L^{\tilde r}(\mu)\big)$. Changing $2t$ to t proves that $p_\theta \in \mathcal{J}_-(\mathcal{T})$, which is a contradiction. We have therefore shown that $p \ge \inf \mathcal{J}_-(\mathcal{T})$. If $p > \inf \mathcal{J}_-(\mathcal{T})$, then $p \in \mathcal{J}_-(\mathcal{T})$ as $p < \tilde r$ and $\mathcal{J}_-(\mathcal{T})$ is an interval. If $p = \inf \mathcal{J}_-(\mathcal{T})$, then $q \in \mathcal{J}_-(\mathcal{T})$, hence $T_t \in \mathcal{O}\big(L^q(\mu) - L^{\tilde r}(\mu)\big)$. As $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ by assumption, we have again $T_{2t} \in \mathcal{O}\big(L^p(\mu) - L^{\tilde r}(\mu)\big)$, hence $p \in \mathcal{J}_-(\mathcal{T})$. We have shown in this case that both p and q belong to $\mathcal{J}_-(\mathcal{T}) \subset \mathcal{J}(\mathcal{T})$.

Case $p \ge \tilde r$**:** This case is similar to the previous one by changing $\inf \mathcal{J}_-(\mathcal{T})$ to $\sup \mathcal{J}_+(\mathcal{T})$ (where the supremum is ∞ if $\mathcal{J}_+(\mathcal{T})$ is unlimited) and arguing on q in place of p.

Case $p < \tilde r < q$**:** By one of the remarks after Definition 2.1, we have that $T_t \in \mathcal{O}\big(L^p(\mu) - L^{\tilde r}(\mu)\big)$ and $T_t \in \mathcal{O}\big(L^{\tilde r}(\mu) - L^q(\mu)\big)$. Hence, by definition, $p \in \mathcal{J}_-(\mathcal{T})$ and $q \in \mathcal{J}_+(\mathcal{T})$.

Let us finish the proof by comparing the interiors of $\mathcal{J}(\mathcal{T})$ and $\widetilde{\mathcal{J}}(\mathcal{T})$. By Theorem 2.3, $\mathcal{J}(\mathcal{T}) \subset \widetilde{\mathcal{J}}(\mathcal{T})$, and the inclusion passes to interiors. Since $\tilde p, \tilde q \in \mathcal{J}(\mathcal{T})$, $\mathcal{O}(\mathcal{T}) \ne \emptyset$. We showed in the previous section that then for each $p \in \operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T})$, $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ for some $q = q(p) > p$. In particular, $p \in \mathcal{J}(\mathcal{T})$ by (4.1). Thus, $\operatorname{Int} \widetilde{\mathcal{J}}(\mathcal{T}) \subset \operatorname{Int} \mathcal{J}(\mathcal{T})$. $\square$

The following result shows that off-diagonal estimates on balls for a semigroup propagate to a sectorial analytic extension with **optimal angle of sectors** provided there is one pair (p_0, p_0) for which one has off-diagonal estimates of balls for the analytic extension.

We consider $\{T_z\}_{z \in \Sigma_\vartheta}$ an analytic semigroup of bounded linear operators on $\mathbb{B}$ with angle $\vartheta < \pi/2$, that is, we assume for $z, z' \in \Sigma_\vartheta = \{\zeta \in \mathbb{C} \setminus \{0\}\,;\, |\arg \zeta| < \vartheta\}$,

$$T_z \in \mathcal{L}(\mathbb{B}); \qquad T_z \circ T_{z'} = T_{z+z'}; \qquad z \in \Sigma_\vartheta \longmapsto T_z \in \mathcal{L}(\mathbb{B}) \text{ is analytic.}$$

We say that $\{T_z\}_{z\in\Sigma_\vartheta} \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ whenever it satisfies the estimates in Definition 2.1 with $|z|$ in place of t. By density, this implies in particular that the semigroup has an analytic extension from Σ_ϑ into $\mathcal{L}(L^r(\mu))$ for $p \le r \le q$.

Recall that $\widetilde{\mathcal{J}}(\mathcal{T})$ denotes the maximal interval of those $p \in [1,\infty]$ for which T_t is bounded on $L^p(\mu)$ uniformly in $t > 0$.

Theorem 4.3. *Let $1 \le p \le p_0 \le q \le \infty$ and ϑ_1 with $0 \le \vartheta_1 < \vartheta$. Assume that $\{T_t\}_{t>0} \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ and that $\{T_z\}_{z\in\Sigma_\vartheta} \in \mathcal{O}\big(L^{p_0}(\mu) - L^{p_0}(\mu)\big)$. Then for any $m \in \mathbb{N}$, $\{z^m \frac{d^m T_z}{dz^m}\}_{z\in\Sigma_{\vartheta_1}} \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$.*

Proof. Assume first $m = 0$. Any $z \in \Sigma_{\vartheta_1}$ has a decomposition $z = s + w + t$ where $w \in \Sigma_\vartheta$, $s, t > 0$ and $|z| \sim |w| \sim s \sim t$, the constants of comparability depending only on ϑ, ϑ_1. Hence, we can write $T_z = T_s \circ T_w \circ T_t$ and use $T_t \in \mathcal{O}\big(L^p(\mu) - L^{p_0}(\mu)\big)$, $T_w \in \mathcal{O}\big(L^{p_0}(\mu) - L^{p_0}(\mu)\big)$ and $T_s \in \mathcal{O}\big(L^{p_0}(\mu) - L^q(\mu)\big)$ together with part (*b*) in Theorem 2.3.

For $m > 0$, we use a third angle ϑ_2 with $\vartheta_1 < \vartheta_2 < \vartheta$. We just showed that $\{T_z\}_{z\in\Sigma_{\vartheta_2}} \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$. To conclude we only have to use Cauchy formulae on circular on circular contours we can compute $\frac{d^m T_z}{dz^m}$ for $z \in \Sigma_{\vartheta_1}$ from T_ζ with $\zeta \in \Sigma_{\vartheta_2}$. $\square$

Note that the assumption on the analytic semigroup is $\mathcal{O}\big(L^{p_0}(\mu) - L^{p_0}(\mu)\big)$ for the same exponent p_0 at both places. In applications, $p_0 = 2$ arises often (see the introduction) but with a weight this exponent is no longer natural.

So far, we were only concerned about the action of the semigroup operator T_t on $L^p(\mu)$ and its off-diagonal estimates. Recall that $\widetilde{\mathcal{J}}(\mathcal{T})$ is the interval of exponents p such that it has an extension to a bounded semigroup to $L^p(\mu)$. To define an infinitesimal generator, it suffices that (the extension to $L^p(\mu)$ of) the semigroup is continuous at 0 for the strong topology in $\mathcal{L}(L^p(\mu))$. As usual, we remove $p = \infty$ from the discussion. However, the off-diagonal estimates play a crucial role.

Proposition 4.4. *Assume that Proposition 4.1 applies and that there is some $r \in \mathcal{J}(\mathcal{T})$, $r \ne \infty$, such that T_t is strongly continuous on $L^r(\mu)$. Then, T_t is strongly continuous on $L^p(\mu)$ for all $p \in \mathcal{J}(\mathcal{T})$ with $p \ne \infty$. In particular, it has an infinitesimal generator on those $L^p(\mu)$.*

Proof. If $p \in \mathcal{J}(\mathcal{T})$ with $p < r$, for f any simple function supported in a ball B we deduce that

$$\Big(\fint_{2B} |T_t f - f|^p \, d\mu\Big)^{\frac{1}{p}} \le \Big(\fint_{2B} |T_t f - f|^r \, d\mu\Big)^{\frac{1}{r}} \longrightarrow 0$$

as $t \to 0$. Next, the off-diagonal estimates on balls imply that

$$\Big(\int_{(2B)^c} |T_t f - f|^p \, d\mu\Big)^{\frac{1}{p}} = \Big(\int_{(2B)^c} |T_t f|^p \, d\mu\Big)^{\frac{1}{p}} \longrightarrow 0$$

as $t \to 0$, using the support of f and Lemma 6.6 below. Then a density argument shows the strong continuity in $L^p(\mu)$.

If $p \in \mathcal{J}(\mathcal{T})$, $r < p < \infty$, then the above applies to the dual semigroup and we can use the well-known fact that on a reflexive space, the dual semigroup of a strongly

continuous bounded semigroup is also strongly continuous (see, *e.g.* [Da1, Chapter 1]). $\square$

Let us turn to weighted off-diagonal estimates. Assume that $\mathcal{T}$ is a semigroup as in Proposition 4.1. Let $w \in A_\infty$. As $(\mathcal{X}, d, w)$ is a space of homogeneous type, we can apply Proposition 4.1 provided we have some off-diagonal estimates to start with. In this case, we can define an interval $\mathcal{J}_w(\mathcal{T})$ characterized as the unique set E in $[1,\infty]$ for which whenever $1 \le p < q \le \infty$ the property $T_t \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ is equivalent to $p, q \in E$. Also, $\widetilde{\mathcal{J}}_w(\mathcal{T})$ is the interval of those $p \in [1,\infty]$ for which T_t is bounded uniformly in t on $L^p(w)$.

Given $1 \le p_0 < q_0 \le \infty$ we define the set

$$\mathcal{W}_w(p_0, q_0) = \big\{p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}\big\}.$$

Corollary 4.5. *Let $1 \le p_0 < q_0 \le \infty$ be such that $(p_0, q_0) \subset \mathcal{J}(\mathcal{T})$ and assume that $\mathcal{W}_w(p_0, q_0) \neq \emptyset$. Then, $\mathcal{W}_w(p_0, q_0) \subset \mathcal{J}_w(\mathcal{T}) \subset \widetilde{\mathcal{J}}_w(\mathcal{T})$ and, consequently,* $\operatorname{Int}\mathcal{J}_w(\mathcal{T}) = \operatorname{Int}\widetilde{\mathcal{J}}_w(\mathcal{T})$*. If, furthermore, $\mathcal{T}$ is strongly continuous on $L^r(\mu)$ for some $r \in (p_0, q_0)$, then $\mathcal{T}$ has an infinitesimal generator in $L^p(w)$ for all $p \in \mathcal{J}_w(\mathcal{T})$, $p \neq \infty$.*

Proof. The first statement is a consequence of Proposition 2.4 and the second of Proposition 4.1 in this context together with the fact shown in [AM1] that if $\mathcal{W}_w(p_0, q_0) \neq \emptyset$ then it is an open interval. Concerning the last statement, by Proposition 4.4 it suffices to check that $\mathcal{T}$ is strongly continuous on $L^p(w)$ for one $p \in \mathcal{J}_w(\mathcal{T})$.

Choose $p \in \mathcal{W}_w(p_0, q_0)$. Then there exists p_1 with $p_0 < p < p_1 < q_0$ and $w \in RH_{\left(\frac{p_1}{p}\right)'}$. Hence,

$$\Big(\fint_B g^p\, dw\Big)^{\frac{1}{p}} \lesssim \Big(\fint_B g^{p_1}\, d\mu\Big)^{\frac{1}{p_1}},$$

for any ball B and positive measurable function g. If we apply this to $g = |T_t f - f|$ for f any simple function supported in a ball B (the hypothesis contains the fact that T_t is defined on $L^{p_1}(\mu)$) and let $t \to 0$, we deduce that

$$\Big(\int_{2\,B} |T_t f - f|^p\, dw\Big)^{\frac{1}{p}} \lesssim \Big(\int_{2\,B} |T_t f - f|^{p_1}\, d\mu\Big)^{\frac{1}{p}{}_1} \longrightarrow 0,$$

where we have used that T_t is strongly continuous on $L^{p_1}(\mu)$ by Proposition 4.4. Next, the off-diagonal estimates on balls for dw imply that

$$\Big(\int_{(2\,B)^c} |T_t f - f|^p\, dw\Big)^{\frac{1}{p}} = \Big(\int_{(2\,B)^c} |T_t f|^p\, dw\Big)^{\frac{1}{p}} \longrightarrow 0, \quad t \to 0,$$

using the support of f and Lemma 6.6 below. Then a density argument shows the strong continuity in $L^p(w)$. $\square$

Remark 4.6. Note that to define $\mathcal{J}_w(\mathcal{T})$, we only need the existence of some pair (p, q) with $p < q$ such that $T_t \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$. Our statement here is a concrete realization of this assumption.

5. A case study

We work in the Euclidean space with the Lebesgue measure. Let $A = A(x)$ be an $n \times n$ matrix of complex and L^∞-valued coefficients defined on $\mathbb{R}^n$. We assume that

this matrix satisfies the following ellipticity (or "accretivity") condition: there exist $0<\lambda\le\Lambda<\infty$ such that

$$\lambda\,|\xi|^2\le \operatorname{Re}A(x)\,\xi\cdot\bar\xi \qquad\text{and}\qquad |A(x)\,\xi\cdot\bar\zeta|\le\Lambda\,|\xi|\,|\zeta|,$$

for all $\xi,\zeta\in\mathbb{C}^n$ and almost every $x\in\mathbb{R}^n$. We have used the notation $\xi\cdot\zeta=\xi_1\,\zeta_1+\cdots+\xi_n\,\zeta_n$ and therefore $\xi\cdot\bar\zeta$ is the usual inner product in $\mathbb{C}^n$. Note that then $A(x)\,\xi\cdot\bar\zeta=\sum_{j,k}a_{j,k}(x)\,\xi_k\,\bar\zeta_j$. Associated with this matrix we define the second order divergence form operator

$$Lf=-\operatorname{div}(A\,\nabla f),$$

which is understood in the standard weak sense by means of a sesquilinear form.

The operator $-L$ generates a C^0-semigroup $\{e^{-t\,L}\}_{t>0}$ of contractions on L^2. We wish to study weighted off-diagonal estimates for $\{e^{-t\,L}\}_{t>0}$ and $\{\sqrt{t}\,\nabla e^{-t\,L}\}_{t>0}$. Before we do so, we recall what is known on unweighted off-diagonal estimates and give some complements.

Remark 5.1. Let us emphasize that on $\mathbb{R}^n$, full off-diagonal estimates are equivalent to off-diagonal estimates on balls in the unweighted situation by Proposition 3.2. This implies in particular that if $e^{-t\,L}\in\mathcal{F}\big(L^p-L^q\big)$ for some $1\le p<q\le\infty$ then (passing to off-diagonal on balls and then going back to full off-diagonal estimates) it follows that $e^{-t\,L}\in\mathcal{F}\big(L^{p_1}-L^{q_1}\big)$ for all $p\le p_1\le q_1\le q$. We will use this fact later.

5.1. **The intervals $\mathcal{J}(L)$ and $\mathcal{K}(L)$.** Define $\widetilde{\mathcal{J}}(L)$ (we change slightly the previous notation to emphasize the dependence on L) as the interval of those exponents $p\in[1,\infty]$ such that $\{e^{-t\,L}\}_{t>0}$ is bounded in $\mathcal{L}(L^p)$.

An almost complete study of L^p-L^q full off-diagonal estimates with $p<q$ for the semigroup has been done in [Aus] (and the exponent θ of the definition must be $\frac12\big(\frac{n}{p}-\frac{n}{q}\big)$). According to Proposition 4.1 and Proposition 3.2, we have the following result.

Proposition 5.2. *There exists a unique subset of* $[1,\infty]$*, denoted by* $\mathcal{J}(L)$*, which is a non empty interval, such that*

$$\forall\,p,q\in[1,\infty],\ p<q\quad\Big(\ e^{-t\,L}\in\mathcal{F}\big(L^p-L^q\big)\Longleftrightarrow p,q\in\mathcal{J}(L)\ \Big).\tag{5.1}$$

Furthermore, $\mathcal{J}(L)\subset\widetilde{\mathcal{J}}(L)$ *and* $\operatorname{Int}\mathcal{J}(L)=\operatorname{Int}\widetilde{\mathcal{J}}(L)$*.*

See Section 4.1 for the meaning of "interior." Write $p_-(L)$ and $p_+(L)$ as the lower and upper bounds in $[1,\infty]$ of $\mathcal{J}(L)$. According to the results proved or cited in [Aus],

$$\mathcal{J}(L)=\widetilde{\mathcal{J}}(L)=[1,\infty],\qquad\text{if } n=1,2,$$

$$p_-(L)<\frac{2n}{n+2}\quad\text{and}\quad p_+(L)>\frac{2n}{n-2},\qquad\text{if } n\ge3.$$

Note that in dimensions $n\ge3$, it is not clear what happens at the endpoints for either boundedness or off-diagonal estimates: can one have boundedness and no off-diagonal estimates? Is $\mathcal{J}(L)$ open in $[1,\infty]$?

Let us turn to the gradient of the semigroup. Define $\widetilde{\mathcal{K}}(L)$ as the interval of those exponents $p\in[1,\infty]$ such that $\{\sqrt{t}\,\nabla e^{-t\,L}\}_{t>0}$ is bounded in $\mathcal{L}(L^p)$. This set has been studied in [Aus]. It is an interval in $[1,\infty]$. If $q_-(L)$ and $q_+(L)$ denote respectively its lower and upper bounds, then it is shown that $q_-(L)=p_-(L)$ and $p_+(L)\ge(q_+(L))^*$

where, given q, its Sobolev exponent q^* is defined as $q^* = n\,q/(n-q)$ if $q < n$ and $q^* = \infty$ otherwise. Also, we always have $q_+(L) > 2$ with $q_+(L) = \infty$ if $n = 1$.

This was proved with the help of full off-diagonal estimates. Define $\mathcal{K}_-(L)$ as the set of all $p \in [1,2]$ such that $\{\sqrt{t}\,\nabla e^{-t\,L}\}_{t>0}$ satisfies $L^p - L^2$ full off-diagonal estimates and $\mathcal{K}_+(L)$ be the set of all $p \in [2,\infty]$ such that $\{\sqrt{t}\,\nabla e^{-t\,L}\}_{t>0}$ satisfies $L^2 - L^p$ full off-diagonal estimates. Set $\mathcal{K}(L) = \mathcal{K}_-(L) \cup \mathcal{K}_+(L)$. This is an interval by interpolation since $2 \in \mathcal{K}(L)$ and it is shown in [Aus] that $\operatorname{Int}\mathcal{K}(L) = \operatorname{Int}\widetilde{\mathcal{K}}(L)$. If $n = 1$, $\mathcal{K}(L) = [1,\infty]$ (see [AMcT]).

We wish to give some further observations, not noticed in [Aus], especially concerning the endpoints of $\mathcal{K}(L)$.

Lemma 5.3. *Let $1 \le p < 2$. The following assertions are equivalent:*

(a) $e^{-t\,L} \in \mathcal{F}\big(L^p - L^2\big)$.

(b) $\sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{F}\big(L^p - L^2\big)$.

(c) $t\,L\,e^{-t\,L} \in \mathcal{F}\big(L^p - L^2\big)$.

Proof. To prove that (a) implies (b), we observe that $\sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{F}\big(L^2 - L^2\big)$ because $2 \in \mathcal{K}(L)$. Hence by composing with (a) and using the semigroup property, we obtain (b).

Similarly, $\sqrt{t}\,e^{-t\,L}\operatorname{div} A \in \mathcal{F}\big(L^2 - L^2\big)$ because of duality and $2 \in \mathcal{K}(L^*)$, and the fact that multiplication by $A(x)$ is bounded on L^2. Hence, from (b), it follows that $\sqrt{t}\,e^{-t\,L}\operatorname{div} A \circ \sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{F}\big(L^p - L^2\big)$. This operator is nothing but $-t\,L\,e^{-2\,t\,L}$ and this proves (c).

Let us assume (c). Pick E,F two closed sets, $f \in L^p \cap L^2$ with support in E and L^p-norm 1 and $g \in L^2$ with support in F and L^2-norm 1. Setting $h(t) = \langle e^{-t\,L} f, g\rangle$, it suffices to prove $|h(t)| \lesssim t^{-\theta}\,e^{-\frac{c\,d^2(E,F)}{t}}$ with $\theta = \frac{1}{2}\,(\frac{n}{p} - \frac{n}{2})$. Observe that our assumption says that $th'(t)$ has such a bound.

First, $\lim_{t\to\infty} h(t) = 0$: this is a consequence of the bounded holomorphic functional calculus for L on L^2 since $z \mapsto e^{-tz}$ converges to 0 uniformly on compact subsets of $\operatorname{Re} z > 0$. Hence, we can write $h(t) = -\int_t^\infty h'(s)\,ds$. Plugging the bound for $sh'(s)$ into this integral yields $|h(t)| \lesssim t^{-\theta}$. This bound suffices when $d^2(E,F) \le t$.

The second case is when $0 < t < d^2(E,F)$. In particular E and F are disjoint. Then, one has $\lim_{s\to 0} h(s) = \langle f, g\rangle = 0$. As $h(t) = \int_0^t h'(s)\,ds$, the bound for $sh'(s)$ easily yields $|h(t)| \lesssim t^{-\theta}\,e^{-\frac{c\,d^2(E,F)}{t}}$. □

Lemma 5.4. *Assume $n \ge 2$. Let $1 \le p < q$ with $q^* < \infty$. If $\sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{F}\big(L^p - L^q\big)$, then $e^{-t\,L} \in \mathcal{F}\big(L^p - L^{q^*}\big)$.*

Proof. By Proposition 3.2, it suffices to show that $e^{-t\,L} \in \mathcal{O}\big(L^p - L^{q^*}\big)$. To this end, we shall need the following form of Sobolev's inequality.

Lemma 5.5. *Let $1 \le q < n$. If $g \in L^1(\mathbb{R}^n)$ with $\nabla g \in L^q(\mathbb{R}^n)$ then $g \in L^{q^*}(\mathbb{R}^n)$ and there is a constant $C > 0$ such that for any ball B,*

$$\Big(\int_{\mathbb{R}^n\setminus B} |g|^{q^*}\,dx\Big)^{\frac{1}{q^*}} \le C\Big(\int_{\mathbb{R}^n\setminus B} |\nabla g|^q\,dx\Big)^{\frac{1}{q}}.$$

The first part of the lemma is non classical but easy: let φ_j be a smooth mollifying sequence and set $g_j=\varphi_j*g$. Then $g_j\in L^1\cap L^\infty(\mathbb{R}^n)$ and $\nabla g_j=\nabla g*\varphi_j\in L^q(\mathbb{R}^n)$, so that in particular $g_j\in W^{1,q}(\mathbb{R}^n)$. Thus Sobolev's inequality on $\mathbb{R}^n$ applies to each g_j and yields

$$\Big(\int_{\mathbb{R}^n}|g_j|^{q^*}\,dx\Big)^{\frac{1}{q^*}}\le C\Big(\int_{\mathbb{R}^n}|\nabla g_j|^q\,dx\Big)^{\frac1q}\le C\Big(\int_{\mathbb{R}^n}|\nabla g|^q\,dx\Big)^{\frac1q}\|\varphi\|_1.$$

Of course, C is independent of j. The conclusion that $g\in L^{q^*}(\mathbb{R}^n)$ follows by applying Fatou's lemma to a subsequence.

We next show the desired Sobolev estimate. It suffices to obtain the desired inequality for B being the unit ball, the general case follows by a change of variable with no change on the constant. Besides, it is enough to assume that $g\in C^1_0(\mathbb{R}^n)$ by density in $W^{1,q}(\mathbb{R}^n)$. Then for any $x\notin B$, one has $g(x)=-\int_0^\infty \pm\partial_j g(x\pm te_j)\,dt$ where e_j is any vector of the canonical basis and the choice of signs depends on the location of x: positive signs when $x_j\ge 0$ and negative signs when $x_j<0$. With this choice of signs, note that if $x\notin B$ we have for all $t\ge 0$, $|x\pm t\,e_j|\ge |x|\ge 1$ and so $x\pm t\,e_j\notin B$. Hence, for all $j=1,\dots,n$, $|g(x)|\le\int_{-\infty}^{+\infty}|\nabla g(x+te_j)|\,\chi_{\mathbb{R}^n\setminus B}(x+te_j)\,dt$. From there, one can follow the standard argument first with $q=1$ and then with other values of q (see, e.g. [Bre]).

We come back to Lemma 5.4, beginning with the proof of (2.4) with respect to dx. Let B be a ball, r its radius and $f\in C_0^\infty(\mathbb{R}^n)$ with support in B. Let $j\ge 2$. Observe that $g=e^{-t\,L}f$ satisfy the hypotheses of Lemma 5.5. Indeed, the full off-diagonal estimates on L^2 and the support of f imply that $\int_{\mathbb{R}^n}|g(x)|^2\,e^{c\,|x-x_B|^2/t}\,dx<\infty$ for some $c>0$ where x_B is the center of B. Hence $g\in L^1(\mathbb{R}^n)$ from Cauchy-Schwarz inequality. Furthermore, $\nabla g\in L^q(\mathbb{R}^n)$ by our assumption. Thus, by Lemma 5.5 and since $\sqrt{t}\,\nabla e^{-t\,L}\in\mathcal{F}\big(L^p-L^q\big)$ we have

$$\begin{aligned}\Big(\fint_{C_j(B)}|e^{-t\,L}f|^{q^*}\,dx\Big)^{\frac{1}{q^*}}&\lesssim (2^j\,r)^{-\frac{n}{q^*}}\Big(\int_{\mathbb{R}^n\setminus 2^j\,B}|\nabla e^{-t\,L}f|^q\,dx\Big)^{\frac1q}\\
&\lesssim (2^j\,r)^{-\frac{n}{q^*}}\sum_{l\ge j}\Big(\int_{C_l(B)}|\nabla e^{-t\,L}f|^q\,dx\Big)^{\frac1q}\\
&\lesssim (2^j\,r)^{-\frac{n}{q^*}}\sum_{l\ge j}t^{-\frac12(\frac np-\frac nq)-\frac12}\,e^{-\frac{c\,4^l\,r^2}{t}}\Big(\int_B|f|^p\,dx\Big)^{\frac1p}\\
&\lesssim 2^{-j\frac{n}{q^*}}\sum_{l\ge j}2^{-l(\frac np-\frac{n}{q^*})}\Big(\frac{2^l\,r}{\sqrt t}\Big)^{\frac np-\frac{n}{q^*}}e^{-\frac{c\,4^l\,r^2}{t}}\Big(\fint_B|f|^p\,dx\Big)^{\frac1p}\\
&\lesssim \Upsilon\Big(\frac{2^j\,r}{\sqrt t}\Big)^{\frac np-\frac{n}{q^*}}e^{-\frac{c\,4^j\,r^2}{t}}\Big(\fint_B|f|^p\,dx\Big)^{\frac1p}.\end{aligned}$$

Hence we obtain (2.4) for $e^{-t\,L}\in\mathcal{O}\big(L^p-L^{q^*}\big)$ with $\theta_1=0$ and $\theta_2=\frac np-\frac{n}{q^*}$.

The proof of (2.2) for $e^{-t\,L}\in\mathcal{O}\big(L^p-L^{q^*}\big)$ is similar using Sobolev's inequality on $\mathbb{R}^n$ only since we do not need a Gaussian term and we obtain the same values for θ_1, θ_2.

It remains to see (2.3) for $e^{-t\,L}\in\mathcal{O}\big(L^p-L^{q^*}\big)$. Let B be a ball, r its radius, $j\ge 2$ and $f\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f\subset C_j(B)$. Since $C_j(B)=2^{j+1}\,B\setminus 2^j\,B$, we can cover

$C_j(B)$ by a finite number of balls $B_{j,k}$ with radii $\frac{5}{8}2^j r$ with centers at distance $\frac{3}{2}2^j r$ from the center of B and the number of balls is a dimensional constant independent of j and B. It is enough to assume that f is also supported in one $B_{j,k}$. Then observe that B is contained in $\mathbb{R}^n \setminus 2B_{j,k}$, hence the preceding argument changing B to $B_{j,k}$ yields (2.3) with θ_1 and θ_2 as above . Details are left to the reader.

In this way we have shown that $e^{-tL} \in \mathcal{O}\big(L^p - L^{q^*}\big)$ and by Remark 5.1 this completes the proof. □

Proposition 5.6. *Assume $n \geq 2$. We have $\mathcal{K}(L) \subset \mathcal{J}(L)$ and $\mathcal{K}(L)$ is characterized by*

$$\forall\, p,q \in [1,\infty],\ p<q\ \Big(\ \sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big) \iff p,q \in \mathcal{K}(L)\ \Big). \tag{5.2}$$

In [Aus], it is only shown that $\operatorname{Int}\mathcal{K}(L) \subset \mathcal{J}(L)$ and the characterization is not considered.

Proof. From Lemma 5.3, $\mathcal{K}_-(L) = \mathcal{J}(L) \cap [1,2]$. From $p_+(L) \geq (q_+(L))^* > q_+(L)$, we have $\mathcal{K}_+(L) \subset \mathcal{J}(L)$. It follows that $\mathcal{K}(L) \subset \mathcal{J}(L)$.

Let us see (5.2). Assume that $p,q \in \mathcal{K}(L)$ with $p<q$. If $p<q\leq 2$ or $2 \leq p < q$, then $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$ as a consequence of $p,q \in \mathcal{K}_-(L)$ or $p,q \in \mathcal{K}_+(L)$ using the equivalence between full off-diagonal estimates and off-diagonal estimates on balls (see Remark 5.1). If $p \leq 2 < q$, then $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^2 - L^q\big)$ and $e^{-tL} \in \mathcal{F}\big(L^p - L^2\big)$ by Lemma 5.3. Hence, by composition and the semigroup property, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$.

We turn to the converse. Let $1 \leq p < q \leq \infty$ with $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$ and let us show that $p,q \in \mathcal{K}(L)$.

Case $2 \leq p < q$**:** We have $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$ and $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^2 - L^2\big)$. Hence, by interpolation, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^{p_\theta} - L^{q_\theta}\big)$ where $1/p_\theta = (1-\theta)/p + \theta/2$, $1/q_\theta = (1-\theta)/q + \theta/2$ and $\theta \in (0,1)$. If $p \notin \mathcal{K}_+(L)$ then $q > \sup \mathcal{K}_+(L)$. We can choose θ such that $p_\theta < \sup\mathcal{K}_+(L) < q_\theta$. Since $\mathcal{K}_+(L) \subset \mathcal{J}(L)$, one has $p_\theta \in \mathcal{J}(L)$, that is, $e^{-tL} \in \mathcal{F}\big(L^2 - L^{p_\theta}\big)$. By composition and the semigroup property, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^2 - L^{q_\theta}\big)$, hence $q_\theta \in \mathcal{K}_+(L)$. This is a contradiction. We have therefore shown that $p \in \mathcal{K}_+(L)$. As we have $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$ by assumption and $e^{-tL} \in \mathcal{F}\big(L^2 - L^p\big)$ since $p \in \mathcal{J}(L)$, by composition and the semigroup property, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^2 - L^q\big)$. Hence $q \in \mathcal{K}_+(L)$.

Case $p < 2 \leq q$**:** Since $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^q\big)$, using the equivalence between off-diagonal estimates on balls and full off-diagonal estimates (see Remark 5.1), we have that $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^2 - L^q\big)$ and $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{F}\big(L^p - L^2\big)$. Hence, $p \in \mathcal{K}_-(L)$ and $q \in \mathcal{K}_+(L)$.

Case $p < q < 2$**:** As $n \geq 2$, we have $q^* < \infty$. Hence, Lemma 5.4 yields in particular $p \in \mathcal{J}(L)$. As $p < 2$, we have $p \in \mathcal{K}_-(L)$ by Lemma 5.3 and since $p<q<2$, $q \in \mathcal{K}_-(L)$ as well. □

Let us finish this section with analyticity issues. For L as above, there exists $\vartheta \in [0, \pi/2)$ depending only on the ellipticity constants such that for all $f \in \mathcal{D}(L)$

$$\big|\arg\langle Lf, f\rangle\big| \le \vartheta.$$

We take the smallest ϑ such that this estimate holds. In this case, one can obtain that L is of type ϑ and its semigroup $\{e^{-t\,L}\}_{t>0}$ has an analytic extension to a complex semigroup $\{e^{-z\,L}\}_{z\in\Sigma_{\pi/2-\vartheta}}$ of contractions on L^2.

Applying Theorem 4.3 with $p_0 = 2$ and Proposition 3.2, one can obtain full off-diagonal estimates for the family $\{(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ in the range $\mathcal{J}(L)$ and a similar type of arguments yields the same thing for the family $\{\sqrt{z}\,\nabla(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ in the range $\mathcal{K}(L)$, where $0 < \mu < \pi/2 - \vartheta$ and $|z|$ replaces t in the estimates. We skip details.

We gather here a particular case for later use in [AM3]. Recall that $\operatorname{Int}\mathcal{J}(L) = \big(p_-(L), p_+(L)\big)$ and $\operatorname{Int}\mathcal{K}(L) = \big(q_-(L), q_+(L)\big)$.

Proposition 5.7. *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$.*

(a) If $p, q \in \big(p_-(L), p_+(L)\big)$ with $p \le q$, then $\{(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$.

(b) If $p, q \in \big(q_-(L), q_+(L)\big)$ with $p \le q$, then $\{\sqrt{z}\,\nabla(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$.

5.2. **The intervals $\mathcal{J}_w(L)$ and $\mathcal{K}_w(L)$.** As a consequence of Proposition 5.7 and Proposition 2.4 we have the following result.

Proposition 5.8. *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$. Let $w \in A_\infty$.*

(a) If $p, q \in \mathcal{W}_w\big(p_-(L), p_+(L)\big)$ with $p \le q$, then $\{(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls and is a bounded set in $\mathcal{L}(L^p(w))$.

(b) If $p, q \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$ with $p \le q$, then $\{\sqrt{z}\,\nabla(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls and is a bounded set in $\mathcal{L}(L^p(w))$.

This statement says that one has some *a priori* knowledge of the intervals were we have weighted off-diagonal estimates on balls. But, they could be larger than this. For a weight w, we let $\widetilde{\mathcal{J}}_w(L)$ and $\widetilde{\mathcal{K}}_w(L)$ be the intervals of exponents $p \in [1, \infty]$ such that $e^{-t\,L}$ and $\sqrt{t}\,\nabla e^{-t\,L}$ respectively are bounded on $L^p(w)$ uniformly in $t > 0$.

Proposition 5.9. *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$. Let $w \in A_\infty$.*

(a) Assume $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \ne \emptyset$. There exists a unique subset of $[1, \infty]$, denoted by $\mathcal{J}_w(L)$, which is an interval containing $\mathcal{W}_w\big(p_-(L), p_+(L)\big)$, such that

$$\forall\, p, q \in [1, \infty],\ p < q \quad \Big(\ e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^q(w)\big) \Longleftrightarrow p, q \in \mathcal{J}_w(L)\ \Big). \tag{5.3}$$

Furthermore, $\mathcal{J}_w(L) \subset \widetilde{\mathcal{J}}_w(L)$ and $\operatorname{Int}\mathcal{J}_w(L) = \operatorname{Int}\widetilde{\mathcal{J}}_w(L)$. Also if $p, q \in \mathcal{J}_w(L)$ with $p \le q$, then $\{(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls and is a bounded set in $\mathcal{L}(L^p(w))$.

(b) Assume $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \ne \emptyset$. There exists a subset of $[1, \infty]$, denoted by $\mathcal{K}_w(L)$, which is an interval containing $\mathcal{W}_w\big(q_-(L), q_+(L)\big)$ with the following properties: if $p, q \in \mathcal{K}_w(L)$ with $p \le q$ then $\sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ and,

conversely, for $p, q \in [1, \infty]$ *with* $p < q$ *and* $p \neq \inf \mathcal{K}_w(L)$, *if* $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ *then* $p, q \in \mathcal{K}_w(L)$. *In particular,* $\mathcal{K}_w(L) \setminus \{\inf \mathcal{K}_w(L)\}$ *is the largest open interval* I *in* $(1, \infty]$ *characterized by*

$$\forall\, p, q \in (1, \infty],\ p < q \quad \Big(\ \sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big) \Longleftrightarrow p, q \in I\ \Big). \tag{5.4}$$

Furthermore, $\mathcal{K}_w(L) \subset \widetilde{\mathcal{K}}_w(L)$ *and* $\operatorname{Int} \mathcal{K}_w(L) = \operatorname{Int} \widetilde{\mathcal{K}}_w(L)$. *Also if* $p, q \in \mathcal{K}_w(L)$ *with* $p \le q$, *then* $\{\sqrt{z}\,\nabla (zL)^m e^{-zL}\}_{z \in \Sigma_\mu}$ *satisfies* $L^p(w) - L^q(w)$ *off-diagonal estimates on balls and is a bounded set in* $\mathcal{L}(L^p(w))$.

(*c*) *Let* $n \ge 2$. *Assume* $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$. *Then* $\mathcal{K}_w(L) \subset \mathcal{J}_w(L)$, $\inf \mathcal{J}_w(L) = \inf \mathcal{K}_w(L)$ *and* $(\sup \mathcal{K}_w(L))^*_w \le \sup \mathcal{J}_w(L)$.

(*d*) *If* $n = 1$, *the intervals* $\mathcal{J}_w(L)$ *and* $\mathcal{K}_w(L)$ *are the same and contain* $(r_w, \infty]$.

We have set $q^*_w = \frac{q\, n\, r_w}{n\, r_w - q}$ when $q < n\, r_w$ and $q^*_w = \infty$ otherwise. Recall that $r_w = \inf\{r \ge 1\,;\, w \in A_r\}$.

Remark 5.10. Let us assume that L has real coefficients. Then, the kernel of e^{-tL} is bounded above and below by Gaussians of the form $C t^{-n/2}\, e^{-\frac{\alpha\, d^2(x,y)}{t}}$ with different constants in each estimate. Hence, for $p \ge 1$ and $w \in A_\infty$, we find that e^{-tL} is bounded on $L^p(w)$ if and only if $w \in A_p$. The sufficiency comes from the upper bound on the kernel. The necessity uses the positivity of the kernel and the doubling condition on w to derive $w \in A_p$. Thus, $\mathcal{J}_w(L) = \{p \in [1, \infty] : w \in A_p\}$. At the same time $\mathcal{W}_w\big(p_-(L), p_+(L)\big) = \mathcal{W}_w(1, \infty) = (r_w, \infty)$. If $w \in A_1$, then one has that $\mathcal{J}_w(L) = [1, \infty]$. If $w \notin A_1$, $\mathcal{J}_w(L) = (r_w, \infty]$. In all cases $\operatorname{Int} \mathcal{J}_w(L) = \mathcal{W}_w\big(p_-(L), p_+(L)\big)$. The positivity of the semigroup makes it in some sense extremal among this class of semigroups (for complex L).

Also $\mathcal{K}_w(L) = (r_w, k_w|$ where $k_w \ge \frac{q_+(L)}{(s_w)'}$ and $s_w = \sup\{s \in (1, \infty]\,;\, w \in RH_s\}$ (whether k_w is in $\mathcal{K}_w(L)$ is not known: we suspect that $\mathcal{K}_w(L)$ is open in $[1, \infty]$).

This also shows that there is no upper bound of $\sup \mathcal{J}_w(L)$ in terms of $\sup \mathcal{K}_w(L)$ as already observed for $w = 1$.

Remark 5.11. We do not know examples where $\mathcal{J}_w(L)$ and $\mathcal{W}_w\big(p_-(L), p_+(L)\big)$ have different endpoints: such examples, if any, must be complex.

Remark 5.12. It seems natural to expect that $\mathcal{J}_w(L)$ and $\mathcal{K}_w(L)$ are included in the set of $r \in [1, \infty]$ such that $w \in A_r$. We are unable to show this.

Also, in part (*b*), we lack of a general argument showing that if $p = \inf \mathcal{K}_w(L) < q \le \inf \mathcal{W}_w\big(q_-(L), q_+(L)\big) = q_-(L) r_w$ and $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ then $p \in \mathcal{K}_w(L)$.

Proof of Proposition 5.9. Part (*a*) follows from Corollary 4.5. For the statement corresponding to the family $\{(z\,L)^m\, e^{-z\,L}\}$ we observe that, given $p, q \in \mathcal{J}_w(L)$ with $p \le q$, there exists $p_0, q_0, r_0 \in \mathcal{J}_w(L)$ so that $p_0 \le p \le q \le q_0$ and $p_0 < r_0 < q_0$ with $r_0 \in \mathcal{W}_w\big(p_-(L), p_+(L)\big)$. By using Proposition 5.8 and Theorem 4.3 it follows that $\{(z\,L)^m\, e^{-z\,L}\} \in \mathcal{O}\big(L^{p_0}(w) - L^{q_0}(w)\big) \subset \mathcal{O}\big(L^p(w) - L^q(w)\big)$.

We next prove part (*d*), part (*c*) and part (*b*) in this order. In fact, the construction of $\mathcal{K}_w(L)$ is given during the proofs of part (*d*) for dimension 1 and part (*c*) for higher dimensions.

Proof of Proposition 5.9, Part (d). We recall that $p_-(L) = q_-(L) = 1$ and $p_+(L) = q_+(L) = \infty$, because the kernels of $e^{-t\,L}$ and of $\sqrt{t}\,\frac{d}{dx}e^{-t\,L}$ are pointwise dominated by Gaussians [AMcT]. Hence, $\mathcal{J}_w(L)$ is the interval of those $p \in [1,\infty]$ such that $e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$ and it contains $(r_w, \infty]$. Define $\mathcal{K}_w(L)$ as the interval of those $p \in [1,\infty]$ such that $\sqrt{t}\,\frac{d}{dx}\,e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$. We show that $\mathcal{J}_w(L) = \mathcal{K}_w(L)$.

Let $p \in \mathcal{J}_w(L)$. As $\sqrt{t}\,\frac{d}{dx}\,e^{-t\,L} \in \mathcal{O}\big(L^\infty(w) - L^\infty(w)\big)$ and $e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$, we have by composition and the semigroup property $\sqrt{t}\,\frac{d}{dx}\,e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$. Hence, $p \in \mathcal{K}_w(L)$.

Conversely, assume $\sqrt{t}\,\frac{d}{dx}\,e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$. Let $q \in \mathbb{R}$ with $\frac{q}{p} > r_w$ so that $w \in A_{\frac{q}{p}}$. Let B be a ball (an interval), r its radius and $f \in C_0^\infty(B)$. Since $e^{-t\,L} f(x)$ vanishes at $\pm\infty$ by the compact support of f and the decay of the kernel of $e^{-t\,L}$, we have for all $x \in \mathbb{R}$,

$$|e^{-t\,L} f(x)| \le \int_x^\infty |(e^{-t\,L} f)'(y)|\,dy \le \int_{\mathbb{R}} |(e^{-t\,L} f)'(y)|\,dy.$$

Now, with $C_l = C_l(B)$, we use $w \in A_{\frac{q}{p}}$ and our assumption, which implies that $\sqrt{t}\,\frac{d}{dx}e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$,

$$\begin{aligned}
\int_{\mathbb{R}} |(e^{-t\,L} f)'(y)|\,dy &\lesssim \sum_{l\ge 1} 2^{l+1}\,r \fint_{C_l} |(e^{-t\,L} f)'(y)|\,dy \\
&\lesssim \sum_{l\ge 1} 2^{l+1}\,r \Big(\fint_{C_l} |(e^{-t\,L} f)'(y)|^p\,dy\Big)^{\frac1p} \\
&\lesssim \sum_{l\ge 1} 2^{l}\,r \Big(\fint_{C_l} |(e^{-t\,L} f)'(y)|^q\,dw(y)\Big)^{\frac1q} \\
&\lesssim r\,t^{-\frac12} \left(\Upsilon\Big(\frac{4\,r}{\sqrt{t}}\Big)^{\theta_2} + \sum_{l\ge 2} 2^{l\,(1+\theta_1)}\,\Upsilon\Big(\frac{2^l\,r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\,4^l\,r^2}{t}}\right) \Big(\fint_B |f|^p\,dw\Big)^{\frac1p} \\
&\lesssim r\,t^{-\frac12}\,\Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\max\{\theta_2, 1+\theta_1\}} \Big(1 + e^{-\frac{c\,r^2}{t}}\Big) \Big(\fint_B |f|^p\,dw\Big)^{\frac1p} \\
&\lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\widetilde{\theta}_2} \Big(\fint_B |f|^p\,dw\Big)^{\frac1p}.
\end{aligned}$$

In particular, this proves (2.2) for $e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$.

Remark that if $x \in C_j$, then one has the more precise estimate

$$|e^{-t\,L} f(x)| \le \int_{\mathbb{R}\setminus 2^j\,B} |(e^{-t\,L} f)'(y)|\,dy.$$

Indeed, it suffices to integrate $(e^{-t\,L} f)'$ from x to $+\infty$ if $x \ge 0$ and from $-\infty$ to x if $x \le 0$. In both cases, the interval of integration is contained in $\mathbb{R} \setminus 2^j\,B$. Hence, the same argument above yields

$$|e^{-t\,L} f(x)| \lesssim \Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\widetilde{\theta}_2} e^{-\frac{\alpha\,4^j\,r^2}{t}} \Big(\fint_B |f|^p\,dw\Big)^{\frac1p},$$

which proves (2.4) for $e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$.

Similarly, assume f supported in $C_j = C_j(B)$. We want to estimate $|e^{-tL}f(x)|$ for $x \in B$. Split C_j into its two connected components, $B_{j,1}, B_{j,2}$, which are intervals of radius $2^{j-1}r$. Observe that B is contained in $\mathbb{R} \setminus 2\,B_{j,k}$ for $k = 1, 2$. Assume that f is supported in $B_{j,1}$ to fix ideas. Hence, for $x \in B$, one has as before

$$|e^{-tL}f(x)| \le \int_{\mathbb{R}\setminus 2\,B_{j,1}} |(e^{-tL}f)'(y)|\,dy.$$

Arguing as above (with $2\,B_{j,1}$ in place of $2^j\,B$) we obtain

$$|e^{-tL}f(x)| \le \Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\widetilde{\theta}_2}\, e^{-\frac{\alpha\,4^j\,r^2}{t}} \Big(\fint_{B_{j,1}} |f|^p\,dw\Big)^{\frac{1}{p}}.$$

One does the same thing when f is supported in $B_{j,2}$. This proves (2.3) for $e^{-tL} \in \mathcal{O}\big(L^p(w) - L^\infty(w)\big)$. Hence, $p \in \mathcal{J}_w(L)$.

Proof of Proposition 5.9, Part (c). We have $n \ge 2$, $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \ne \emptyset$ and we know that this is an open interval. Pick $\tilde{r} \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$ and set

$$\begin{aligned}
\mathcal{K}_{-,w}(L) &= \{p \in [1,\tilde{r}]\,;\, \sqrt{t}\,\nabla\, e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde{r}}(w)\big)\},\\
\mathcal{K}_{+,w}(L) &= \{p \in [\tilde{r},\infty]\,;\, \sqrt{t}\,\nabla\, e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^p(w)\big)\},\\
\mathcal{K}_w(L) &= \mathcal{K}_{-,w}(L) \cup \mathcal{K}_{+,w}(L).
\end{aligned}$$

By construction, $\mathcal{K}_w(L)$ contains $\mathcal{W}_w\big(q_-(L), q_+(L)\big)$ and it is clearly an interval.

We need the following lemmas whose proofs are given below:

Lemma 5.13. *Let $1 \le p < \tilde{r}$. The following assertions are equivalent:*

(i) $e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde{r}}(w)\big)$.

(ii) $\sqrt{t}\,\nabla\, e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde{r}}(w)\big)$.

Lemma 5.14. *Assume $\tilde{r} < p \le \infty$ and $\sqrt{t}\,\nabla\, e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^p(w)\big)$. Then for $\tilde{r} \le q < p^*_w$, we have $e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^q(w)\big)$.*

Note that Lemma 5.13 yields that $\inf \mathcal{J}_w(L) = \inf \mathcal{K}_w(L)$ and $\mathcal{K}_{-,w}(L) \subset \mathcal{J}_w(L)$. On the other hand, Lemma 5.14 implies that $(\sup \mathcal{K}_w(L))^*_w \le \sup \mathcal{J}_w(L)$ and so $\mathcal{K}_{+,w}(L) \subset \mathcal{J}_w(L)$. This proves part (c).

Proof of Lemma 5.13. As $\tilde{r} \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$, we have $\sqrt{t}\,\nabla\, e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^{\tilde{r}}(w)\big)$. Hence, by composition and the semigroup property, we deduce that (i) implies (ii).

For the converse, we cannot follow the route of Lemma 5.3 so we use similar ideas as in Lemma 5.4. We introduce some auxiliary exponents. Since $\tilde{r} \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$, there exist p_1, q_1 such that $q_-(L) < p_1 < \tilde{r} < q_1 < q_+(L)$ and $w \in A_{\frac{\tilde{r}}{p_1}} \cap RH_{\left(\frac{q_1}{\tilde{r}}\right)'}$. Note that $q_+(L) > (q_-(L))^*$: indeed if $n = 2$ then $q_-(L) = 1$ and $q_+(L) > 2$ whereas if $n \ge 3$, $q_-(L) < \frac{2n}{n+2}$ and $q_+(L) > 2$. Thus one can choose $p_1 < n$ and q_1 so that $p_1^* = \frac{n\,p_1}{n-p_1} \le q_1$.

We begin the proof of (i) with (2.4). Let B be a ball, r its radius and $f \in C_0^\infty(\mathbb{R}^n)$ with support in B. Let $j \ge 2$ and $C_j = C_j(B)$. Observe that $g = e^{-tL}f$ satisfy

the hypotheses of Lemma 5.5 with $q = p_1$. We use that $w \in RH_{(\frac{q_1}{\tilde r})'}$ and $e^{-(t/2)\,L} \in \mathcal{F}\big(L^{p_1^*} - L^{q_1}\big)$ —because $p_-(L) = q_-(L) < p_1 < p_1^* \le q_1 < q_+(L) \le p_+(L)$— and Sobolev's inequality on $\mathbb{R}^n$ and on $\mathbb{R}^n \setminus 2^{j-1}\,B$ (see Lemma 5.5):

$$
\begin{aligned}
\Big(\fint_{C_j} |e^{-t\,L} f|^{\tilde r}\,dw\Big)^{\frac{1}{\tilde r}} &\lesssim \Big(\fint_{C_j} |e^{-t\,L} f|^{q_1}\,dx\Big)^{\frac{1}{q_1}} \\
&\lesssim (2^j\,r)^{-\frac{n}{q_1}}\, t^{-\frac12\,(\frac{n}{p_1^*}-\frac{n}{q_1})} \bigg[e^{-\frac{c\,4^j\,r^2}{t}} \Big(\int_{2^{j-1}\,B} |e^{-(t/2)\,L} f|^{p_1^*}\,dx\Big)^{\frac{1}{p_1^*}} \\
&\qquad\qquad + \Big(\int_{\mathbb{R}^n\setminus 2^{j-1}\,B} |e^{-(t/2)\,L} f|^{p_1^*}\,dx\Big)^{\frac{1}{p_1^*}}\bigg] \\
&\lesssim (2^j\,r)^{-\frac{n}{q_1}}\, t^{-\frac12\,(\frac{n}{p_1^*}-\frac{n}{q_1})} \bigg[e^{-\frac{c\,4^j\,r^2}{t}} \Big(\int_{\mathbb{R}^n} |\nabla e^{-(t/2)\,L} f|^{p_1}\,dx\Big)^{\frac{1}{p_1}} \\
&\qquad\qquad + \Big(\int_{\mathbb{R}^n\setminus 2^{j-1}\,B} |\nabla e^{-(t/2)\,L} f|^{p_1}\,dx\Big)^{\frac{1}{p_1}}\bigg].
\end{aligned}
$$

From $w \in A_{\frac{\tilde r}{p_1}}$ and our assumption (ii), we have

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^n} |\nabla e^{-(t/2)\,L} f|^{p_1}\,dx\Big)^{\frac{1}{p_1}} &\lesssim \sum_{l\ge1} (2^{l+1}\,r)^{\frac{n}{p_1}} \Big(\fint_{C_l} |\nabla e^{-(t/2)\,L} f|^{p_1}\,dx\Big)^{\frac{1}{p_1}} \\
&\lesssim\ t^{-\frac12} \sum_{l\ge1} (2^{l+1}\,r)^{\frac{n}{p_1}} \Big(\fint_{C_l} |\sqrt{t}\,\nabla e^{-(t/2)\,L} f|^{\tilde r}\,dw\Big)^{\frac{1}{\tilde r}} \\
&\lesssim\ r^{\frac{n}{p_1}}\, t^{-\frac12} \bigg(\Upsilon\Big(\frac{4\,r}{\sqrt{t}}\Big)^{\theta_2} + \sum_{l\ge2} 2^{l\,(\frac{n}{p_1}+\theta_1)}\, \Upsilon\Big(\frac{2^l\,r}{\sqrt t}\Big)^{\theta_2} e^{-\frac{c\,4^l\,r^2}{t}}\bigg) \Big(\fint_B |f|^p\,dw\Big)^{\frac1p} \\
&\lesssim\ r^{\frac{n}{p_1}}\, t^{-\frac12}\, \Upsilon\Big(\frac{r}{\sqrt t}\Big)^{\max\{\theta_2,\frac{n}{p_1}+\theta_1\}} \Big(1 + e^{-\frac{c\,r^2}{t}}\Big) \Big(\fint_B |f|^p\,dw\Big)^{\frac1p} \\
&\lesssim\ r^{\frac{n}{p_1}}\, t^{-\frac12} \Upsilon\Big(\frac{r}{\sqrt t}\Big)^{\widetilde\theta_2} \Big(\fint_B |f|^p\,dw\Big)^{\frac1p}.
\end{aligned}
\tag{5.5}
$$

The integral on $\mathbb{R}^n \setminus 2^{j-1}B$ is analyzed similarly when $j \ge 3$ with a summation over $l \ge j-1$. If $j = 2$, then the integral on $C_1 = 4B$ is replaced by one on $\widehat{C}_1 = 4B \setminus 2B$ whose contribution is of the same order than the one on C_2. Hence,

$$
\Big(\int_{\mathbb{R}^n\setminus 2^{j-1}\,B} |\nabla e^{-(t/2)\,L} f|^{p_1}\,dx\Big)^{\frac{1}{p_1}} \lesssim (2^j\,r)^{\frac{n}{p_1}}\, t^{-\frac12}\, \Upsilon\Big(\frac{2^j\,r}{\sqrt t}\Big)^{\widetilde\theta_2} e^{-\frac{c\,4^j\,r^2}{t}} \Big(\fint_B |f|^p\,dw\Big)^{\frac1p}.
$$

All together after rearranging the terms yields

$$
\Big(\fint_{C_j} |e^{-t\,L} f|^{\tilde r}\,dw\Big)^{\frac{1}{\tilde r}} \lesssim 2^{j\,\widetilde\theta_2}\, \Upsilon\Big(\frac{2^j\,r}{\sqrt t}\Big)^{\widetilde\theta_2+\frac{n}{p_1}-\frac{n}{q_1}} e^{-\frac{c\,4^j\,r^2}{t}} \Big(\fint_B |f|^p\,dw\Big)^{\frac1p},
$$

which is (2.4) for $e^{-t\,L} \in \mathcal{O}\big(L^p(w) - L^{\tilde r}(w)\big)$.

The proof of (2.2) for $e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde r}(w)\big)$ is similar using only Sobolev's inequality on $\mathbb{R}^n$ since we do not need a Gaussian term:

$$\Big(\fint_B |e^{-tL} f|^{\tilde r}\, dw\Big)^{\frac{1}{\tilde r}} \lesssim \Big(\fint_B |e^{-tL} f|^{q_1}\, dx\Big)^{\frac{1}{q_1}} \lesssim r^{-\frac{n}{q_1}}\, t^{-\frac12(\frac{n}{p_1^*}-\frac{n}{q_1})} \Big(\int_{\mathbb{R}^n} |e^{-(t/2)L} f|^{p_1^*}\, dx\Big)^{\frac{1}{p_1^*}}$$
$$\lesssim r^{-\frac{n}{q_1}}\, t^{-\frac12(\frac{n}{p_1^*}-\frac{n}{q_1})} \Big(\int_{\mathbb{R}^n} |\nabla e^{-(t/2)L} f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}.$$

¿From here we conclude the desired estimate as in (5.5).

It remains to prove (2.3) for $e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde r}(w)\big)$. Let B be a ball, r its radius, $j \ge 2$ and $f \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subset C_j = C_j(B)$. Since $C_j = 2^{j+1} B \setminus 2^j B$, we can cover C_j by a finite number of balls $B_{j,k}$ with radii $\frac58 2^j r$, with centers at distance $\frac32 2^j r$ from the center of B and the number of balls is a dimensional constant independent of j and B. It is enough to assume that f is also supported in one $B_{j,k}$. Using $w \in RH_{\left(\frac{q_1}{\tilde r}\right)'}$ and $e^{-(t/2)L} \in \mathcal{F}\big(L^{p_1^*} - L^{q_1}\big)$,

$$\Big(\fint_B |e^{-tL} f|^{\tilde r}\, dw\Big)^{\frac{1}{\tilde r}} \lesssim \Big(\fint_B |e^{-tL} f|^{q_1}\, dx\Big)^{\frac{1}{q_1}}$$
$$\lesssim r^{-\frac{n}{q_1}}\, t^{-\frac12(\frac{n}{p_1^*}-\frac{n}{q_1})} \bigg[e^{-\frac{c\,4^j r^2}{t}} \Big(\int_{\mathbb{R}^n \setminus 2^{j-2} B} |e^{-(t/2)L} f|^{p_1^*}\, dx\Big)^{\frac{1}{p_1^*}}$$
$$+ \Big(\int_{2^{j-2} B} |e^{-(t/2)L} f|^{p_1^*}\, dx\Big)^{\frac{1}{p_1^*}} \bigg].$$

We use Sobolev's inequality in $\mathbb{R}^n$ and split $\mathbb{R}^n$ according to the sets $C_l(2^{j+1} B)$, $l \ge 1$. Then $w \in A_{\frac{\tilde r}{p_1}}$ and (ii) yield as before

$$\Big(\int_{\mathbb{R}^n \setminus 2^{j-2} B} |e^{-(t/2)L} f|^{p_1^*}\, dx\Big)^{\frac{1}{p_1^*}} \lesssim \Big(\int_{\mathbb{R}^n} |\nabla e^{-(t/2)L} f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}$$
$$\lesssim (2^j r)^{\frac{n}{p_1}}\, t^{-\frac12}\, \Upsilon\Big(\frac{2^j r}{\sqrt t}\Big)^{\widetilde\theta_2} \Big(\fint_{2^{j+1} B} |f|^p\, dw\Big)^{\frac1p}$$

with $\widetilde\theta_2 = \max\{\theta_2, \frac{n}{p_1} + \theta_1\}$. Next, observe that $2^{j-2} B$ is contained in $\mathbb{R}^n \setminus 2B_{j,k}$, hence by Lemma 5.5,

$$\Big(\int_{2^{j-2} B} |e^{-(t/2)L} f|^{p_1^*}\, dx\Big)^{\frac{1}{p_1^*}} \lesssim \Big(\int_{\mathbb{R}^n \setminus 2B_{j,k}} |\nabla e^{-(t/2)L} f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}.$$

Using a splitting of $\mathbb{R}^n \setminus 2B_{j,k}$ with the rings $2^{l+1} B_{j,k} \setminus 2^l B_{j,k}$ for $l \ge 1$, $w \in A_{\frac{\tilde r}{p_1}}$ and (ii) together with Lemma 6.5 give us

$$\Big(\int_{\mathbb{R}^n \setminus 2B_{j,k}} |\nabla e^{-(t/2)L} f|^{p_1}\, dx\Big)^{\frac{1}{p_1}} \lesssim (2^j r)^{\frac{n}{p_1}}\, t^{-\frac12}\, \Upsilon\Big(\frac{2^j r}{\sqrt t}\Big)^{\widetilde\theta_2} e^{-\frac{c\,4^j r^2}{t}} \Big(\fint_{B_{j,k}} |f|^p\, dw\Big)^{\frac1p}.$$

Gathering our estimates, we deduce that

$$\Big(\fint_B |e^{-tL} f|^{\tilde r}\, dw\Big)^{\frac{1}{\tilde r}} \lesssim \Upsilon\Big(\frac{2^j r}{\sqrt t}\Big)^{\widetilde\theta_2 + \frac{n}{p_1} - \frac{n}{q_1}} e^{-\frac{c\,4^j r^2}{t}} \Big(\fint_{C_j} |f|^p\, dw\Big)^{\frac1p}$$

whenever f is supported in $C_j \cap B_{j,k}$. This gives us (2.3) for $e^{-tL} \in \mathcal{O}\big(L^p(w) - L^{\tilde{r}}(w)\big)$. $\square$

Proof of Lemma 5.14. We know that $\tilde{r} \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$. Hence, $w \in A_{\frac{\tilde{r}}{q_-(L)}} \subset A_{\tilde{r}} \subset A_p$. Furthermore, for all $r > r_w$, all balls B and Borel subsets E of B,

$$\frac{|E|}{|B|} \lesssim \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{r}}.$$

Let $q < \infty$ with $\frac{1}{r} \geq \frac{n}{p} - \frac{n}{q}$. Using [FPW, Corollary 3.2], we have an $L^p(w) - L^q(w)$ Poincaré inequality:

$$\Big(\fint_B |g - g_B|^q \, dw\Big)^{\frac{1}{q}} \lesssim r_B \Big(\fint_B |\nabla g|^p \, dw\Big)^{\frac{1}{p}},$$

for all any B and Lipschitz function g where g_B stands for the w-average of g on B. Since convolution with a C_0^∞ function defines bounded map on $L^r(w)$ when $w \in A_r$, an approximation argument via mollifiers shows the validity of this inequality if $g \in L^{\tilde{r}}(w)$ such that $\nabla g \in L^p(w)$.

We begin with (2.2) for $e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^q(w)\big)$. Since $\tilde{r} \in \mathcal{W}_w\big(q_-(L), q_+(L)\big) \subset \mathcal{W}_w\big(p_-(L), p_+(L)\big)$, $e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^{\tilde{r}}(w)\big)$. The matter is to improve integrability. Let B be a ball, r its radius and $f \in C_0^\infty(\mathbb{R}^n)$ with support in B. Observe that the Poincaré inequality above applies on B to $g = e^{-tL} f$ since we know that $g \in L^{\tilde{r}}(w)$, $\nabla g \in L^p(w)$ from $e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^{\tilde{r}}(w)\big)$, our assumption $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^{\tilde{r}}(w) - L^p(w)\big)$ and Lemma 6.6. Hence

$$\begin{aligned}
\Big(\fint_B |e^{-tL} f|^q \, dw\Big)^{\frac{1}{q}} &\lesssim \fint_B |e^{-tL} f| \, dw + r \Big(\fint_B |\nabla e^{-tL} f|^p \, dw\Big)^{\frac{1}{p}}. \\
&\lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\theta_2} \Big(\fint_B |f|^{\tilde{r}} \, dw\Big)^{\frac{1}{\tilde{r}}}.
\end{aligned}$$

This proves (2.2).

To prove (2.4), we take B and f as before. Let $j \geq 2$ and cover $C_j = C_j(B)$ by a finite number of balls $B_{j,k}$ with radii $\frac{5}{8} 2^j r$ and centers at distance $\frac{3}{2} 2^j r$ from the center of B. For each ball $B_{j,k}$, we apply the same argument and obtain (2.4) using the hypothesis with C_j replaced by each $B_{j,k}$. It suffices to add all the estimates to conclude.

To prove (2.3), we apply the same argument as for (2.2) but with f now supported in $C_j(B)$ for $j \geq 2$. Easy details are skipped. $\square$

Proof of Proposition 5.9, Part (b). In parts (c) and (d), we defined a set $\mathcal{K}_w(L)$ which is an interval in $[1, \infty]$ containing $\mathcal{W}_w\big(q_-(L), q_+(L)\big)$. The proof that $p, q \in \mathcal{K}_w(L)$ with $p \leq q$ implies $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ is entirely similar to that of Proposition 5.6 for $\mathcal{K}(L)$ replacing 2 by $\tilde{r}$, full off-diagonal estimates by off-diagonal estimates on balls and using Lemma 5.13 in place of Lemma 5.3.

Conversely assuming $p, q \in [1, \infty]$ with $p < q$ and $p \neq \inf \mathcal{K}_w(L)$ and $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ we conclude that $p, q \in \mathcal{K}_w(L)$ as in Proposition 5.6 for $\mathcal{K}(L)$ except when $p < q \leq \tilde{r}$. For this situation, we argue as follows: As $p \neq \inf \mathcal{K}_w(L)$ we have two cases. The first one is $p > \inf \mathcal{K}_w(L)$, which yields p, q in the interval $\mathcal{K}_w(L)$. The second one is $p < \inf \mathcal{K}_w(L)$. Interpolating $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$

with $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^{\tilde p}(w) - L^{\tilde q}(w)\big)$ for any $\tilde p, \tilde q \in \mathcal{K}_w(L)$ with $\tilde p < \tilde q$, one can find a pair p_θ, q_θ with $p_\theta < \inf \mathcal{K}_w(L)$ and $q_\theta \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$ such that $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}\big(L^{p_\theta}(w) - L^{q_\theta}(w)\big)$. Lemma 5.13 holds with q_θ in place of $\tilde r$ as $q_\theta \in \mathcal{W}_w\big(q_-(L), q_+(L)\big)$ and thus $p_\theta \in \mathcal{J}_w(L)$. This leads to a contradiction since $p_\theta < \inf \mathcal{K}_w(L) = \inf \mathcal{J}_w(L)$. Hence, this second case does not happen.

The rest of the proof of (b) is easy: That $\operatorname{Int} \mathcal{K}_w(L) = \operatorname{Int} \widetilde{\mathcal{K}}_w(L)$ is a consequence of the discussion in Section 4.1 and the previous characterization (see also the last part of the proof of Proposition 4.1). The extension of the off-diagonal estimates to the analytic family follows that done for the semigroup. The $L^p(w)$ boundedness follows from Theorem 2.3, part (a). We skip further details. □

We conclude this discussion with a word on infinitesimal generators.

Corollary 5.15. *Assume $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$. For $p \in \mathcal{J}_w(L)$ and $p \neq \infty$, the extension to $L^p(w)$ of $\{e^{-tL}\}_{t>0}$ has an infinitesimal generator which is an operator of type ϑ in $L^p(w)$.*

Proof. This is a consequence of Corollary 4.5 and Proposition 5.9, noting that by construction $\{e^{-tL}\}_{t>0}$ is strongly continuous on L^2 and $2 \in (p_-(L), p_+(L))$. The fact that the infinitesimal generator is of type ϑ comes from the holomorphic extension of the semigroup on $\Sigma_{\pi/2-\vartheta}$ for $p \in \mathcal{J}_w(L)$. □

Remark 5.16. The inclusion $\mathcal{K}_w(L) \subset \mathcal{J}_w(L)$ implies that if $p \in \mathcal{K}_w(L)$ with $p \neq \infty$, the domain of the $L^p(w)$-infinitesimal generator is contained in the space $\{f \in L^p(w)\,;\, \nabla f \in L^p(w)\}$ (the gradient is defined in the distributional sense).

6. Proofs of the main results

6.1. **Proof of Proposition 2.2.** Assume first that $K_t(x,y)$ is given with the desired properties. Fix $t>0$. Let B be a ball, r its radius and z its center. Let $f \in L^1(\mu)$ with support in B. Then for almost every $x \in B$,

$$|T_t f(x)| \le \frac{C}{\mu(B(x,\sqrt{t}))} \int_B |f|\,d\mu \le \frac{C\,\mu(B)}{\mu(B(x,\sqrt{t}))} \Big(\fint_B |f|\,d\mu \Big).$$

The doubling condition yields that $\mu(B) \approx \mu(B(x,r))$. If $r \le \sqrt{t}$ then $\mu(B) \lesssim \mu(B(x,\sqrt{t})$. Otherwise $r \ge \sqrt{t}$, the doubling condition implies

$$\mu(B) \approx \mu(B(x,r)) \lesssim \Big(\frac{r}{\sqrt{t}}\Big)^D \mu(B(x,\sqrt{t})),$$

and (2.2) holds with $\theta_2 = D$, the doubling exponent of μ. Similarly, (2.3) and (2.4) hold with $\theta_1 = 0$ and $\theta_2 = D$. Hence $T_t \in \mathcal{O}\big(L^1(\mu) - L^\infty(\mu)\big)$.

Conversely, assume $T_t \in \mathcal{O}\big(L^1(\mu) - L^\infty(\mu)\big)$. Fix $t>0$. It follows in particular from (2.2) that for any ball B and any $f, g \in L^1$ with support in B

$$\int_B |g(x)|\,|T_t f(x)|\,d\mu(x) \le \frac{C}{\mu(B)}\,\Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\theta_2} \|f\|_1 \|g\|_1.$$

Hence, there exists $K_{t,B} \in L^\infty(B \times B)$ such that

$$\int_B g(x)\,T_t f(x)\,d\mu(x) = \int_B \int_B g(x) K_{t,B}(x,y)\,f(y)\,d\mu(y)\,d\mu(x).$$

It is easy to show that $K_{t,B}(x,y)=K_{t,B'}(x,y)$ almost everywhere on $B\times B\cap B'\times B'$ so that we may define $K_t\in L^\infty_{\rm loc}(\mathcal{X}\times\mathcal{X})$ which agrees almost everywhere with $K_{t,B}$ on $B\times B$. Fix a Lebesgue point (x_0,y_0) of K_t. Assume that $d(x_0,y_0)<\sqrt{t}$. Fix $B=B(x_0,\sqrt{t})$ so that $x_0,y_0\in B$. Then, apply the formula above and let f, g approximate Dirac masses at y_0,x_0 (more precisely, we use Lebesgue differentiation) to obtain

$$|K_t(x_0,y_0)|\le \frac{C}{\mu(B)}=\frac{C}{\mu(B(x_0,\sqrt{t}))}.$$

If $d(x_0,y_0)\ge\sqrt{t}$, then we choose $r=d(x_0,y_0)/6$, and for $f\in L^1(\mu)$ with support in $B=B(y_0,r)$ and $g\in L^1(\mu)$ with support in $B'=B(x_0,r)$, we have (embed B and B' in a larger ball B'' on which the formula for $K_{t,B''}$ is used)

$$\int_{B'} g(x)\,T_tf(x)\,d\mu(x)=\int_{B'}\int_B g(x)K_t(x,y)\,f(y)\,d\mu(y)\,d\mu(x).$$

Since $B'\subset C_2(B)$, we may apply (2.4) with $j=2$ and by letting f and g approximate Dirac masses as before, we obtain

$$|K_t(x_0,y_0)|\le\frac{C}{\mu(B)}\Big(\frac{d(x_0,y_0)}{\sqrt{t}}\Big)^{\theta_2}e^{-\frac{c\,d^2(x_0,y_0)}{t}}.$$

But $d(x_0,y_0)\ge\sqrt{t}$ implies that we can absorb the θ_2 power by the Gaussian factor and also that $\mu(B(x_0,\sqrt{t}))\le\mu(B(x_0,d(x_0,y_0)))\lesssim\mu(B(x_0,r))=\mu(B)$ as μ is doubling.

6.2. **Proof of Theorem 2.3: Part** (a)**.** We need the following basic facts about spaces of homogeneous type. Indeed, the following property was used originally to define those spaces, see [CW].

Lemma 6.1. *There exists $N\in\mathbb{N}$ depending on C_0 in* (2.1)*, such that, for every $j\ge 1$, any ball B contains at most N^j points $\{x_k\}_k$ such that $d(x_{k_1},x_{k_2})>r_B/2^j$.*

We also recall the following well-known covering lemma whose proof is left to the reader (note that the covering family has to be countable since in any fixed ball the number of $r/2$-separated points is finite by the previous result).

Lemma 6.2. *Given $r>0$ there exists a sequence $\{x_k\}_k\subset\mathcal{X}$ so that $d(x_{k_1},x_{k_2})>r/2$ for all $x_{k_1}\neq x_{k_2}$ and $\mathcal{X}=\bigcup_k B(x_k,r)$.*

We can now establish (a) in Theorem 2.3. We use Lemma 6.2 with $r=\sqrt{t}$ and write $B_k=B(x_k,\sqrt{t})$. Then, if $1\le p<\infty$,

$$\begin{aligned}
\|T_tf\|_{L^p(\mu)}^p &\le \sum_k\int_{B_k}|T_tf|^p\,d\mu\le\sum_k\bigg(\sum_{j=1}^\infty\Big(\int_{B_k}|T_t(\chi_{C_j(B_k)}\,f)|^p\,d\mu\Big)^{\frac1p}\bigg)^p\\
&\lesssim \sum_k\bigg(\sum_{j=1}^\infty 2^{j\,\theta_1}\,\Upsilon\big(2^j\big)^{\theta_2}\,e^{-c\,4^j}\,\Big(\frac{\mu(B_k)}{\mu(2^{j+1}B_k)}\Big)^{\frac1p}\Big(\int_{C_j(B_k)}|f|^p\,d\mu\Big)^{\frac1p}\bigg)^p\\
&\lesssim \sum_k\bigg(\sum_{j=1}^\infty e^{-c\,4^j}\int_{C_j(B_k)}|f|^p\,d\mu\bigg)\bigg(\sum_{j=1}^\infty 2^{j\,(\theta_1+\theta_2)\,p'}\,e^{-c\,4^j}\bigg)^{\frac{p}{p'}}\\
&\lesssim \sum_{j=1}^\infty e^{-c\,4^j}\int_{\mathcal{X}}|f|^p\sum_k\chi_{C_j(B_k)}\,d\mu\le\sum_{j=1}^\infty e^{-c\,4^j}\,N^{j+3}\int_{\mathcal{X}}|f|^p\,d\mu
\end{aligned}$$

$$\lesssim \int_{\mathcal{X}} |f|^p \, d\mu,$$

where we have used that for any $j \ge 1$ we have $\sum_k \chi_{C_j(B_k)}(x) \le N^{j+3}$. Indeed, for a fixed $x \in \mathcal{X}$, there exists k_0 such that $x \in B_{k_0}$. Then, by Lemma 6.1,

$$\sum_k \chi_{C_j(B_k)}(x) \le \#\{k : x \in 2^{j+1} B_k\} \le \#\big\{k : x_k \in B(x_{k_0}, 2^{j+2} r)\big\} \le N^{j+3},$$

since $d(x_k, x_j) > r/2 = (2^{j+2} r)/2^{j+3}$. The modification for $p = \infty$ is left to the reader.

6.3. **Proof of Theorem 2.3: Part** (b)**.** We next show that the definition of off-diagonal estimates on balls is stable under composition.

To prove this we need the following auxiliary results whose proofs are postponed until the end of this subsection.

Lemma 6.3. *Let $s > 0$, $\alpha \ge 0$ and $\beta > 0$ with $\alpha \ne \beta$. Then, if $0 < c' < c$,*

$$\sum_{k=0}^{\infty} 2^{k\,\alpha}\, \Upsilon\big(2^k s\big)^{\beta}\, e^{-c\,4^k s^2} \lesssim \Upsilon(s)^{\max\{\alpha,\beta\}}\, e^{-c'\,s^2}.$$

Remark 6.4. We have assumed $\alpha \ne \beta$ in order to get explicit exponents. If

$\alpha = \beta$ the same estimate remains true with the power of $\Upsilon(s)$ being $\alpha + \varepsilon$, for any $\varepsilon > 0$, in place of α. For cleanness and shortness, we will use this lemma several times assuming that the powers are different, if this is not the case the final power has to be slightly enlarged for the estimate to be correct.

Lemma 6.5. *If $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ with exponents θ_1 and θ_2, then for any ball B with radius r we have*

$$\Big(\fint_B |T_t(\chi_{\widehat{C}_1(B)}\, f)|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim \Upsilon\Big(\frac{2\,r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\,4\,r^2}{t}} \Big(\fint_{\widehat{C}_1(B)} |f|^p \, d\mu\Big)^{\frac{1}{p}}$$

and

$$\Big(\fint_{\widehat{C}_1(B)} |T_t(\chi_B\, f)|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim \Upsilon\Big(\frac{2\,r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\,4\,r^2}{t}} \Big(\fint_B |f|^p \, d\mu\Big)^{\frac{1}{p}},$$

that is, T_t satisfies the last two estimates in Definition 2.1 with $j = 1$ and $\widehat{C}_1(B)$ in place of $C_j(B)$.

Lemma 6.6. *If $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ with parameters θ_1, θ_2, c then for $0 < c' < c$, for any ball B with radius r and for every $j \ge 1$ we have*

$$\Big(\fint_B |T_t(\chi_{(2^j B)^c}\, f)|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim 2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\max\{\theta_1,\theta_2\}} e^{-\frac{c'\,4^j\,r^2}{t}} \Big(\fint_{(2^j B)^c} |f|^p \, d\mu\Big)^{\frac{1}{p}}$$

and

$$\Big(\fint_{(2^j B)^c} |T_t(\chi_B\, f)|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim 2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j\,r}{\sqrt{t}}\Big)^{\max\{\theta_1 + D/q,\theta_2\}} e^{-\frac{c'\,4^j\,r^2}{t}} \Big(\fint_B |f|^p \, d\mu\Big)^{\frac{1}{p}}.$$

Once we have stated these auxiliary results we can proceed to establish Theorem 2.3, Part (b).

Proof of Theorem 2.3: Part (*b*). We start with (2.2) and assume that $\operatorname{supp} f \subset B$. Write $\lambda = r_B/\sqrt{t}$. Note that we have

$$\begin{aligned}\Big(\fint_B |T_t(S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} &\le \Big(\fint_B |T_t(\chi_{2B}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} + \Big(\fint_B |T_t(\chi_{(2B)^c}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} \\ &= I + II.\end{aligned}$$

Then, since $\operatorname{supp} f \subset B \subset 2B$ we have

$$\begin{aligned} I &\lesssim \Big(\fint_{2B} |T_t(\chi_{2B}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} \lesssim \Upsilon(2\lambda)^{\theta_2} \Big(\fint_{2B} |S_t f|^q \, d\mu\Big)^{\frac{1}{q}} \\ &\lesssim \Upsilon(2\lambda)^{\theta_2+\gamma_2} \Big(\fint_{2B} |f|^q \, d\mu\Big)^{\frac{1}{q}} \lesssim \Upsilon(\lambda)^{\theta_2+\gamma_2} \Big(\fint_{B} |f|^p \, d\mu\Big)^{\frac{1}{p}}.\end{aligned}$$

On the other hand, applying Lemma 6.6 twice we have

$$\begin{aligned} II &\lesssim 2^{\theta_1}\, \Upsilon(2\lambda)^{\max\{\theta_1,\theta_2\}}\, e^{-c'\,4\,\lambda^2} \Big(\fint_{(2B)^c} |S_t f|^q \, d\mu\Big)^{\frac{1}{q}} \\ &\lesssim 2^{\theta_1+\gamma_1}\, \Upsilon(2\lambda)^{\max\{\theta_1,\theta_2\}+\max\{\gamma_1+D/q,\gamma_2\}}\, e^{-c'\,4\,\lambda^2} \Big(\fint_{2B} |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &\lesssim \Upsilon(\lambda)^{\max\{\theta_1,\theta_2\}+\max\{\gamma_1+D/q,\gamma_2\}} \Big(\fint_{B} |f|^p \, d\mu\Big)^{\frac{1}{p}}.\end{aligned}$$

Collecting the bounds for I and II we obtain the desired estimate.

Next, we consider (2.3). Let f be supported on $C_j(B)$ with $j \ge 2$. Let us set $\lambda = 2^j\, r_B/\sqrt{t}$. We first split the integral as follows

$$\begin{aligned}\Big(\fint_B |T_t(S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} &\le \Big(\fint_B |T_t(\chi_{2^{j-1}B}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} + \Big(\fint_B |T_t(\chi_{(2^{j-1}B)^c}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} \\ &= I + II.\end{aligned}$$

For I we write $\widetilde{B} = 2^{j-1} B$ which has radius $r_{\widetilde{B}} = 2^{j-1}\, r_B$. Thus, $C_j(B) = 4\,\widetilde{B} \setminus 2\,\widetilde{B} = \widehat{C}_1(\widetilde{B})$ and so by Lemma 6.5 we have

$$\begin{aligned} I &\le \Big(\frac{\mu(2^{j-1}B)}{\mu(B)}\Big)^{\frac{1}{r}} \Big(\fint_{\widetilde{B}} |T_t(\chi_{\widetilde{B}}\, S_t f)|^r \, d\mu\Big)^{\frac{1}{r}} \\ &\lesssim 2^{j\,D/r}\, \Upsilon\Big(\frac{2\, r_{\widetilde{B}}}{\sqrt{t}}\Big)^{\theta_2} \Big(\fint_{\widetilde{B}} |S_t(\chi_{\widehat{C}_1(\widetilde{B})}\, f)|^q \, d\mu\Big)^{\frac{1}{q}} \\ &\lesssim 2^{j\,D/r}\, \Upsilon(\lambda)^{\theta_2}\, \Upsilon\Big(\frac{r_{\widetilde{B}}}{\sqrt{t}}\Big)^{\gamma_2} e^{-\frac{c\,4\,r_{\widetilde{B}}^2}{t}} \Big(\fint_{\widehat{C}_1(\widetilde{B})} |f|^p \, d\mu\Big)^{\frac{1}{p}} \\ &\lesssim 2^{j\,D/r}\, \Upsilon(\lambda)^{\theta_2+\gamma_2}\, e^{-c\,\lambda^2} \Big(\fint_{C_j(B)} |f|^p \, d\mu\Big)^{\frac{1}{p}}.\end{aligned}$$

On the other hand, by Lemma 6.6

$$\begin{aligned} II &\lesssim 2^{(j-1)\,\theta_1}\, \Upsilon\Big(\frac{2^{j-1}\, r_B}{\sqrt{t}}\Big)^{\max\{\theta_1,\theta_2\}} e^{-\frac{c'\,4^{j-1}\,r_B^2}{t}} \Big(\fint_{(2^{j-1}B)^c} |S_t f|^q \, d\mu\Big)^{\frac{1}{q}} \\ &\lesssim 2^{j\,\theta_1}\, \Upsilon(\lambda)^{\max\{\theta_1,\theta_2\}}\, e^{-c'\,\lambda^2} \Big(\fint_{(2^{j-1}B)^c} |S_t f|^q \, d\mu\Big)^{\frac{1}{q}}.\end{aligned}$$

Besides,

$$\Big(\fint_{(2^{j-1}B)^c}|S_t f|^q\,d\mu\Big)^{\frac1q}\lesssim\Big(\fint_{2^{j+2}B}|S_t f|^q\,d\mu\Big)^{\frac1q}+\Big(\fint_{(2^{j+2}B)^c}|S_t f|^q\,d\mu\Big)^{\frac1q}=II_1+II_2.$$

For II_1 we observe that

$$\begin{aligned}II_1 &= \Big(\fint_{2^{j+2}B}|S_t(\chi_{2^{j+2}B}\,f)|^q\,d\mu\Big)^{\frac1q}\lesssim\Upsilon\Big(\frac{r_{2^{j+2}B}}{\sqrt t}\Big)^{\gamma_2}\ \Big(\fint_{2^{j+2}B}|f|^p\,d\mu\Big)^{\frac1p}\\ &\lesssim \Upsilon(\lambda)^{\gamma_2}\ \Big(\fint_{C_j(B)}|f|^p\,d\mu\Big)^{\frac1p}.\end{aligned}$$

On the other hand for II_2 we set $\widetilde B=2^{j+1}B$ and so its radius is $r_{\widetilde B}=2^{j+1}r_B$. Thus, $(2^{j+2}B)^c=(2\,\widetilde B)^c$ and $r_{\widetilde B}/\sqrt t=2\lambda$. By Lemma 6.6 we have

$$\begin{aligned}II_2 &= \Big(\fint_{(2\,\widetilde B)^c}|S_t(f\ \chi_{\widetilde B})|^q\,d\mu\Big)^{\frac1q}\lesssim 2^{\gamma_1}\Upsilon(4\,\lambda)^{\max\{\gamma_1+D/q,\gamma_2\}}\ e^{-c'\,\lambda^2}\ \Big(\fint_{\widetilde B}|f|^p\,d\mu\Big)^{\frac1p}\\ &\lesssim \Upsilon(\lambda)^{\max\{\gamma_1+D/q,\gamma_2\}}\ e^{-c'\,\lambda^2}\ \Big(\fint_{C_j(B)}|f|^p\,d\mu\Big)^{\frac1p}.\end{aligned}$$

Collecting the bounds for I, II_1 and II_2 we obtain the desired estimate.

Finally, we show (2.4). We take $\operatorname{supp} f\subset B$ and $j\ge 2$. Let us set $\lambda=2^j\,r_B/\sqrt t$. We proceed as follows

$$\begin{aligned}\Big(\fint_{C_j(B)}|T_t(S_t f)|^r\,d\mu\Big)^{\frac1r} &\le \Big(\fint_{C_j(B)}|T_t(\chi_{2^{j-1}B}\ S_t f)|^r\,d\mu\Big)^{\frac1r}\\ &\quad+\Big(\fint_{C_j(B)}|T_t(\chi_{(2^{j-1}B)^c}\ S_t f)|^r\,d\mu\Big)^{\frac1r}=I+II.\end{aligned}$$

For I we write $\widetilde B=2^{j-1}B$ which has radius $r_{\widetilde B}=2^{j-1}r$. Thus, $C_j(B)=4\,\widetilde B\setminus 2\,\widetilde B=\widehat C_1(\widetilde B)$ and $r_{\widetilde B}/\sqrt t=\lambda/2$, so Lemma 6.5 yields

$$\begin{aligned}I &= \Big(\fint_{\widehat C_1(\widetilde B)}|T_t(\chi_{\widetilde B}\ S_t f)|^r\,d\mu\Big)^{\frac1r}\lesssim\Upsilon(\lambda)^{\theta_2}\ \ e^{-c\,\lambda^2}\Big(\fint_{\widetilde B}|S_t f|^q\,d\mu\Big)^{\frac1q}\\ &= \Upsilon(\lambda)^{\theta_2}\ e^{-c\,\lambda^2}\Big(\fint_{\widetilde B}|S_t(f\,\chi_{\widetilde B})|^q\,d\mu\Big)^{\frac1q}\lesssim\Upsilon(\lambda)^{\theta_2}\ e^{-c\,\lambda^2}\,\Upsilon(\lambda)^{\gamma_2}\Big(\fint_{\widetilde B}|f|^p\,d\mu\Big)^{\frac1p}\\ &\lesssim \Upsilon(\lambda)^{\theta_2+\gamma_2}\ e^{-c\,\lambda^2}\Big(\fint_B|f|^p\,d\mu\Big)^{\frac1p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}II &\lesssim \Big(\fint_{2^{j+2}B}|T_t(\chi_{2^{j+2}B\setminus 2^{j-1}B}\ S_t f)|^r\,d\mu\Big)^{\frac1r}+\Big(\fint_{2^{j+1}B}|T_t(\chi_{(2^{j+2}B)^c}\ S_t f)|^r\,d\mu\Big)^{\frac1r}\\ &= II_1+II_2.\end{aligned}$$

For II_1 we use Lemma 6.6:

$$\begin{aligned}II_1 &= \Big(\fint_{2^{j+2}B}|T_t(\chi_{2^{j+2}B}\,(\chi_{(2^{j-1}B)^c}\ S_t f))|^r\,d\mu\Big)^{\frac1r}\\ &\lesssim \Upsilon\Big(\frac{r_{2^{j+2}B}}{\sqrt t}\Big)^{\theta_2}\ \Big(\fint_{2^{j+2}B}\chi_{(2^{j-1}B)^c}\ |S_t f|^q\,d\mu\Big)^{\frac1q}\end{aligned}$$

$$\lesssim \Upsilon(\lambda)^{\theta_2}\, 2^{(j-1)\gamma_1}\, \Upsilon\Big(\frac{2^{j-1}\, r_B}{\sqrt{t}}\Big)^{\max\{\gamma_1+D/q,\gamma_2,\}}\, e^{-\frac{c\,4^{j-1}\, r_B^2}{t}}\, \Big(\fint_B |f|^p\, d\mu\Big)^{\frac1p}$$

$$\lesssim 2^{j\,\gamma_1}\, \Upsilon(\lambda)^{\theta_2+\max\{\gamma_1+D/q,\gamma_2,\}}\, e^{-c\,\lambda^2}\, \Big(\fint_B |f|^p\, d\mu\Big)^{\frac1p}.$$

On the other hand for II_2 we set $\widetilde{B} = 2^{j+1}\, B$ and so its radius is $r_{\widetilde{B}} = 2^{j+1}\, r$. Thus, $(2^{j+2}\, B)^c = (2\, \widetilde{B})^c$ and $r_{\widetilde{B}}/\sqrt{t} = 2\lambda$, so by Lemma 6.6 we have

$$II_2 = \Big(\fint_{\widetilde{B}} |T_t(\chi_{(2\,\widetilde{B})^c}\; S_t f)|^r\, d\mu\Big)^{\frac1r}$$

$$\lesssim 2^{\theta_1}\, \Upsilon\Big(\frac{2\, r_{\widetilde{B}}}{\sqrt{t}}\Big)^{\max\{\theta_1,\theta_2\}}\, e^{-c\,\lambda^2}\, \Big(\fint_{(2\,\widetilde{B})^c} |S_t f|^q\, d\mu\Big)^{\frac1q}$$

$$\lesssim \Upsilon(\lambda)^{\max\{\theta_1,\theta_2\}}\, e^{-c\,\lambda^2}\, \Big(\fint_{(2^{j+2}\, B)^c} |S_t f|^q\, d\mu\Big)^{\frac1q}$$

$$\lesssim \Upsilon(\lambda)^{\max\{\theta_1,\theta_2\}}\, e^{-c\,\lambda^2}\, 2^{(j+2)\,\gamma_1}\, \Upsilon\Big(\frac{2^{j+2}\, r_B}{\sqrt{t}}\Big)^{\max\{\gamma_1+D/q,\gamma_2\}}\, e^{-\frac{c\,4^{j+2}\, r_B^2}{t}}\, \Big(\fint_{\widetilde{B}} |f|^p\, d\mu\Big)^{\frac1p}$$

$$\lesssim 2^{j\,\gamma_1}\, \Upsilon(\lambda)^{\max\{\theta_1,\theta_2\}+\max\{\gamma_1+D/q,\gamma_2\}}\, e^{-c\,\lambda^2}\, \Big(\fint_B |f|^p\, d\mu\Big)^{\frac1p}.$$

Collecting the bounds for I, II_1 and II_2 we obtain the desired estimate. □

Proof of Lemma 6.3. If $s \ge 1$, since $s^\beta\, e^{-c\,4^k\, s^2} \lesssim e^{-c'\, s^2} \cdot e^{-c''\, 4^k}$ for some $c'' > 0$, we have

$$\sum_{k=0}^\infty 2^{k\,\alpha}\, \Upsilon\big(2^k\, s\big)^\beta\, e^{-c\,4^k\, s^2} \lesssim e^{-c'\, s^2} \sum_{k=0}^\infty 2^{k\,(\alpha+\beta)}\, e^{-c''\, 4^k} \lesssim e^{-c'\, s^2} \lesssim \Upsilon(s)^{\max\{\alpha,\beta\}}\, e^{-c'\, s^2}.$$

If $0 < s < 1$ then there is $k_0 \in \mathbb{N}$ such that $2^{-k_0} \le s < 2^{-k_0+1}$. We obtain

$$\sum_{k=0}^\infty 2^{k\,\alpha}\, \Upsilon\big(2^k\, s\big)^\beta\, e^{-c\,4^k\, s^2} \lesssim \sum_{k=0}^\infty 2^{k\,\alpha}\, \Upsilon\big(2^{k-k_0}\big)^\beta\, e^{-c\,4^{k-k_0}}$$

$$= \sum_{k=0}^{k_0} 2^{k\,\alpha}\, \Upsilon\big(2^{k-k_0}\big)^\beta\, e^{-c\,4^{k-k_0}} + \sum_{k=k_0+1}^\infty 2^{k\,\alpha}\, \Upsilon\big(2^{k-k_0}\big)^\beta\, e^{-c\,4^{k-k_0}} = I + II.$$

Then, as $\alpha \ne \beta$,

$$I \le \sum_{k=0}^{k_0} 2^{k\,\alpha}\, 2^{-(k-k_0)\,\beta} \lesssim 2^{k_0\, \max\{\alpha,\beta\}} \lesssim s^{-\max\{\alpha,\beta\}}.$$

On the other hand, as $\alpha + \beta > 0$

$$II \le \sum_{k=k_0+1}^\infty 2^{k\,\alpha}\, 2^{(k-k_0)\,\beta}\, e^{-c\,4^{k-k_0}} \lesssim 2^{k_0\,\alpha} \sum_{k=1}^\infty 2^{k\,(\alpha+\beta)}\, e^{-c\,4^k} \lesssim 2^{k_0\,\alpha} \lesssim s^{-\max\{\alpha,\beta\}}.$$

Thus,

$$\sum_{k=0}^\infty 2^{k\,\alpha}\, \Upsilon\big(2^k\, s\big)^\beta\, e^{-c\,4^k\, s^2} \le s^{-\max\{\alpha,\beta\}} \lesssim \Upsilon(s)^{\max\{\alpha,\beta\}}\, e^{-c'\, s^2}.$$

□

Proof of Lemma 6.5. By Lemma 6.1, given B we can construct a sequence $\{x_k\}_{k=1}^K \subset B$ with $K \le N^3$ such that $d(x_k, x_j) > r_B/8$ for $j \neq k$ and with the property that for all $x \in B$ we have some k for which $d(x, x_k) \le r_B/8$ (this means that we cannot pick more x_k's). Write $B_k = B(x_k, r_B/4)$ and note that $B \subset \bigcup_{k=1}^K B_k$. Besides, $\widehat{C}_1(B) \subset 2^5\, B_k \setminus 2^2\, B_k = C_2(B_k) \cup C_3(B_k) \cup C_4(B_k)$ for each k. Let f be supported on $\widehat{C}_1(B)$. Then, for each k, $f = \sum_{j=2}^4 f_{j,k}$ where $f_{j,k} = f\, \chi_{C_j(B_k)}$. As supp $f_{j,k} \subset C_j(B_k)$,

$$\begin{aligned}
\Big(\fint_B |T_t f|^q\, d\mu\Big)^{\frac1q} &\le \sum_{k=1}^K \Big(\frac{\mu(B_k)}{\mu(B)}\Big)^{\frac1q} \Big(\fint_{B_k} |T_t f|^q\, d\mu\Big)^{\frac1q} \lesssim \sum_{k=1}^K \sum_{j=2}^4 \Big(\fint_{B_k} |T_t f_{j,k}|^q\, d\mu\Big)^{\frac1q} \\
&\lesssim \sum_{k=1}^K \sum_{j=2}^4 2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j\, r(B_k)}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\, 4^j\, r(B_k)^2}{t}} \Big(\fint_{C_j(B_k)} |f_{j,k}|^p\, d\mu\Big)^{\frac1p} \\
&\lesssim \Upsilon\Big(\frac{2\, r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\, 4\, r^2}{t}} \Big(\fint_{\widehat{C}_1(B)} |f|^p\, d\mu\Big)^{\frac1p}.
\end{aligned}$$

where we have used that $\mu(2^{j+1}B_k) \approx \mu(4B) \approx \mu(B)$ for $j = 2, 3, 4$ and $1 \le k \le K$.

On the other hand if supp $f \subset B$ we have that $f = \sum_{k=1}^N f_k$ where $f_k = f\, \chi_{E_k}$ with $E_k \subset B_k$ and the sets E_k are pairwise disjoint (for instance, we can take $E_1 = B_1$, $E_2 = B_2 \setminus E_1$, ...). Then, as supp $f_k \subset B_k$

$$\begin{aligned}
\Big(\fint_{\widehat{C}_1(B)} |T_t f|^q\, d\mu\Big)^{\frac1q} &\le \sum_{k=1}^K \sum_{j=2}^4 \Big(\frac{\mu(2^{j+1}\, B_k)}{\mu(4\, B)}\Big)^{\frac1q} \Big(\fint_{C_j(B_k)} |T_t f_k|^q\, d\mu\Big)^{\frac1q} \\
&\lesssim \sum_{k=1}^K \sum_{j=2}^4 2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j\, r(B_k)}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\, 4^j\, r(B_k)^2}{t}} \Big(\fint_{B_k} |f_k|^p\, d\mu\Big)^{\frac1p} \\
&\lesssim \Upsilon\Big(\frac{2\, r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\, 4\, r^2}{t}} \Big(\fint_B |f|^p\, d\mu\Big)^{\frac1p}.
\end{aligned}$$

□

Proof of Lemma 6.6. Suppose first that $j \ge 2$. Let us write $\lambda = 2^j\, r/\sqrt{t}$. Using that $T_t \in \mathcal{O}\big(L^p(\mu) - L^q(\mu)\big)$ we have

$$\begin{aligned}
\Big(\fint_B |T_t(\chi_{(2^j\, B)^c}\, f)|^q\, d\mu\Big)^{\frac1q} &\le \sum_{k \ge j} \Big(\fint_B |T_t(\chi_{C_k(B)}\, f)|^q\, d\mu\Big)^{\frac1q} \\
&\lesssim \sum_{k \ge j} 2^{k\,\theta_1} \Upsilon\Big(\frac{2^k\, r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\, 4^k\, r^2}{t}} \Big(\fint_{C_k(B)} |f|^p\, d\mu\Big)^{\frac1p} \\
&\le 2^{j\,\theta_1} \sum_{k=0}^\infty 2^{k\,\theta_1} \Upsilon\big(2^k\, \lambda\big)^{\theta_2} e^{-c\, 4^k\, \lambda^2} \Big(\fint_{(2^j\, B)^c} |f|^p\, d\mu\Big)^{\frac1p} \\
&\lesssim 2^{j\,\theta_1}\, \Upsilon(\lambda)^{\max\{\theta_1, \theta_2\}} e^{-c'\, \lambda^2} \Big(\fint_{(2^j\, B)^c} |f|^p\, d\mu\Big)^{\frac1p},
\end{aligned}$$

where we have used Lemma 6.3 (the power $\max\{\theta_1, \theta_2\}$ is correct whenever $\theta_1 \neq \theta_2$, see Remark 6.4 otherwise).

When $j=1$, the argument is exactly the same but for the term $k=j=1$ on which we use Lemma 6.5 in place of Definition 2.1.

On the other hand, assume that $j\geq 2$ (and the other case is done as just explained). Then, since μ is doubling, we have

$$\begin{aligned}\Big(\fint_{(2^j B)^c} |T_t(\chi_B\, f)|^q\,d\mu\Big)^{\frac1q} &\le \sum_{k\ge j}\left(\frac{\mu(2^{k+1}B)}{\mu(2^{j+1}B)}\right)^{\frac1q}\Big(\fint_{C_k(B)} |T_t(\chi_B\, f)|^q\,d\mu\Big)^{\frac1q}\\ &\lesssim \sum_{k\ge j} 2^{(k-j)\,D/q}\,2^{k\,\theta_1}\,\Upsilon\Big(\frac{2^k\,r}{\sqrt t}\Big)^{\theta_2}\ e^{-\frac{c\,4^k\,r^2}{t}}\Big(\fint_B |f|^p\,d\mu\Big)^{\frac1p}\\ &= 2^{j\,\theta_1}\sum_{k=0}^{\infty} 2^{k\,(\theta_1+D/q)}\,\Upsilon\big(2^k\,\lambda\big)^{\theta_2}\ e^{-c\,4^k\,\lambda^2}\Big(\fint_B |f|^p\,d\mu\Big)^{\frac1p}\\ &\lesssim 2^{j\,\theta_1}\,\Upsilon\big(2^k\,\lambda\big)^{\max\{\theta_1+D/q,\theta_2\}}\ e^{-c'\,4^j\,\lambda^2}\Big(\fint_{(2^j B)^c} |f|^p\,d\mu\Big)^{\frac1p},\end{aligned}$$

where, as before, we have used Lemma 6.3 (here, one needs $\theta_1+D/q\neq\theta_2$, see Remark 6.4 otherwise). $\square$

6.4. **Proof of Proposition 2.4.** We fix $w\in A_{\frac{p}{p_0}}\cap RH_{(\frac{q_0}{q})'}$ and by Proposition A.1, (iii) and (iv), there exist p_1, q_1 with $p_0<p_1<p\le q<q_1<q_0$ such that $w\in A_{\frac{p}{p_1}}\cap RH_{(\frac{q_1}{q})'}$. It is well-known that

$$w\in A_{\frac{p}{p_1}}\iff \Big(\fint_B g^{p_1}\,d\mu\Big)^{\frac{1}{p_1}}\lesssim\Big(\fint_B g^{p}\,dw\Big)^{\frac1p}\tag{6.1}$$

$$w\in RH_{(\frac{q_1}{q})'}\iff \Big(\fint_B g^{q}\,dw\Big)^{\frac1q}\lesssim\Big(\fint_B g^{q_1}\,d\mu\Big)^{\frac{1}{q_1}},\tag{6.2}$$

where in the right hand sides g runs over the set of non-negative measurable functions and B runs over the set of balls. Thus, using (6.2), $T_t\in\mathcal{O}\big(L^{p_1}(\mu)-L^{q_1}(\mu)\big)$ and (6.1) we have

$$\begin{aligned}\Big(\fint_B |T_t(\chi_B\, f)|^q\,dw\Big)^{\frac1q}&\lesssim\Big(\fint_B |T_t(\chi_B\, f)|^{q_1}\,d\mu\Big)^{\frac{1}{q_1}}\lesssim\Upsilon\Big(\frac{r}{\sqrt t}\Big)^{\theta_2}\Big(\fint_B |f|^{p_1}\,d\mu\Big)^{\frac{1}{p_1}}\\ &\lesssim\Upsilon\Big(\frac{r}{\sqrt t}\Big)^{\theta_2}\Big(\fint_B |f|^{p}\,dw\Big)^{\frac1p}.\end{aligned}$$

This shows (2.2). The same can be done to derive (2.3) and (2.4) and this completes the proof.

6.5. **Proof of Propositions 3.2 and 3.3.**

Proof of Proposition 3.2. We prove (a). That $T_t\in\mathcal{F}\big(L^p(\mu)-L^q(\mu)\big)$ implies $T_t\in\mathcal{O}\big(L^p(\mu)-L^q(\mu)\big)$ is easy by specializing (3.1) to balls and annuli and using

$$\mu(B)^{\frac1p-\frac1q}\lesssim r^{\frac np-\frac nq}.$$

from the polynomial upper bound of the volume.

We turn to (b). Assume $q<\infty$. The argument mimics that of Theorem 2.3, part (a). Let E,F be two closed sets and $t>0$. Let f be supported in E. We first assume

that $t < (d(E,F)/16)^2$. Pick a collection of balls $B_k = B(x_k, r)$ as in Lemma 6.2 with $r = d(E,F)/16$. Observe that if $x \in F$ and $y \in E$ then $d(x,y) \geq d(E,F) = 16\,r$. Hence, if $x \in B_k$ then $y \notin 4B_k$, so $y \in C_j(B_k)$ for some $j \geq 2$. In what follows the summation in k is restricted to those balls B_k so that $F \cap B_k \neq \emptyset$. Using that $\operatorname{supp} f \subset E$ and (2.3), we have

$$\|T_t f\|_{L^q(F,\mu)}^q \leq \sum_k \int_{B_k} |T_t f|^q \, d\mu \leq \sum_k \bigg(\sum_{j=2}^{\infty} \Big(\int_{B_k} |T_t(\chi_{C_j(B_k)}\, f)|^q \, d\mu \Big)^{\frac{1}{q}} \bigg)^q$$

$$\lesssim \sum_k \bigg(\sum_{j=2}^{\infty} 2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\,4^j\,r^2}{t}} \frac{\mu(B_k)^{\frac{1}{q}}}{\mu(2^{j+1}\,B_k)^{\frac{1}{p}}} \Big(\int_{C_j(B_k)} |f|^p \, d\mu \Big)^{\frac{1}{p}} \bigg)^q.$$

Next, by the polynomial lower bound of the volume, $p \leq q$ and $r > \sqrt{t}$, we have

$$\frac{\mu(B_k)^{\frac{1}{q}}}{\mu(2^{j+1}\,B_k)^{\frac{1}{p}}} \leq \frac{\mu(B_k)^{\frac{1}{q}}}{\mu(B_k)^{\frac{1}{p}}} \lesssim r^{\frac{n}{q}-\frac{n}{p}} \leq t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}.$$

Also, note that

$$2^{j\,\theta_1}\, \Upsilon\Big(\frac{2^j r}{\sqrt{t}}\Big)^{\theta_2} e^{-\frac{c\,4^j\,r^2}{t}} \lesssim e^{-c'\,4^j}\, e^{-\frac{c''\,r^2}{t}}$$

for some $c', c'' > 0$. Thus,

$$\begin{aligned}
\|T_t f\|_{L^q(F,\mu)}^q &\lesssim \Big(t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}\, e^{-\frac{c''r^2}{t}} \Big)^q \sum_k \bigg(\sum_{j=2}^{\infty} e^{-c'\,4^j} \Big(\int_{C_j(B_k)} |f|^p \, d\mu \Big)^{\frac{1}{p}} \bigg)^q \\
&\lesssim \Big(t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}\, e^{-\frac{c''r^2}{t}} \Big)^q \sum_k \bigg(\sum_{j=2}^{\infty} e^{-c'\,4^j} \int_{C_j(B_k)} |f|^p \, d\mu \bigg)^{\frac{q}{p}} \\
&\leq \Big(t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}\, e^{-\frac{c''r^2}{t}} \Big)^q \bigg(\sum_k \sum_{j=2}^{\infty} e^{-c'\,4^j} \int_{C_j(B_k)} |f|^p \, d\mu \bigg)^{\frac{q}{p}} \\
&= \Big(t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}\, e^{-\frac{c''r^2}{t}} \Big)^q \bigg(\int_E \sum_{j=2}^{\infty} \sum_k e^{-c'\,4^j}\, \chi_{C_j(B_k)}\, |f|^p \, d\mu \bigg)^{\frac{q}{p}}
\end{aligned}$$

where we used the fact that $\frac{q}{p} \geq 1$. We conclude as in Section 6.2 using

$$\sum_{j=2}^{\infty} \sum_k e^{-c'\,4^j}\, \chi_{C_j(B_k)} \leq \sum_{j=2}^{\infty} N^{j+3} e^{-c'\,4^j} \leq C < \infty.$$

In the case where $t \geq (d(E,F)/16)^2$, then we argue as before with $r = \sqrt{t}$. This time, we have to incorporate the terms with $j = 1$ and use also (2.2) with $4B_k$. We obtain

$$\|T_t f\|_{L^q(F,\mu)} \lesssim t^{-\frac{1}{2}\,(\frac{n}{p}-\frac{n}{q})}\, \|f\|_{L^p(E,\mu)}.$$

This proves the result when $q < \infty$. The modification for $q = \infty$ is left to the reader. □

Proof of Proposition 3.3. Assume first that T_t satisfies $L^p(\mu) - L^q(\mu)$ full off-diagonal estimates of type φ. Let us see for example, how to obtain (2.2) for $T_t \in \mathcal{O}\big(L^p(\mu) -$

$L^q(\mu)\big)$, the other estimates being similar. We specialize (3.1) with $t^{-\theta}$ replaced by $\varphi\big(\sqrt{t}\big)^{\frac{1}{q}-\frac{1}{p}}$ to $E=F=B$. Then, using $\mu(B)\sim\varphi(r)$, we obtain

$$\Big(\fint_B |T_t f|^q\,d\mu\Big)^{\frac{1}{q}} \lesssim \left(\frac{\varphi(r)}{\varphi\big(\sqrt{t}\big)}\right)^{\frac{1}{p}-\frac{1}{q}} \Big(\fint_B |f|^p\,d\mu\Big)^{\frac{1}{p}}$$

and we observe that since φ is non decreasing and μ is doubling,

$$\frac{\varphi(r)}{\varphi\big(\sqrt{t}\big)} \lesssim \max\Big\{\Big(\frac{r}{\sqrt{t}}\Big)^D,1\Big\} \lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^D$$

where D is the doubling order of μ.

We turn to the converse. The argument is the same as the one above. The only change is in the inequality when $t<(d(E,F)/16)^2=r^2$, and it reads

$$\frac{\mu(B_k)^{\frac{1}{q}}}{\mu(2^{j+1}\,B_k)^{\frac{1}{p}}} \le \frac{\mu(B_k)^{\frac{1}{q}}}{\mu(B_k)^{\frac{1}{p}}} \sim \varphi(r)^{\frac{1}{q}-\frac{1}{p}} \le \varphi\big(\sqrt{t}\big)^{\frac{1}{q}-\frac{1}{p}}$$

as φ is non-decreasing $\sqrt{t}\le r$ and $p\le q$. Further details are left to the reader. $\square$

6.6. **Proof of Proposition 3.6.** Fix a ball B. Set $r=r_B$ and $\lambda=r/\sqrt{t}$. By (iii) and (iv) in Proposition A.1, one can select p_1,q_1 with $p_0<p_1<p\le q<q_1<q_0$ and $w\in A_{\frac{p}{p_1}}\cap RH_{(\frac{q_1}{q})'}$.

Using (vii) and $(viii)$ in Proposition A.1, we have that $w^{(\frac{q_1}{q})'}\in A_\alpha$ with $\alpha=1+(\frac{p}{p_1}-1)(\frac{q_1}{q})'$ and

$$\Big(\int_{(2B)^c} w^{(\frac{q_1}{q})'}\Big(\frac{|x-x_B|}{r}\Big)^{-n\alpha}dx\Big)^{1/(\frac{q_1}{q})'} \lesssim \Big(\int_B w^{(\frac{q_1}{q})'}(x)dx\Big)^{1/(\frac{q_1}{q})'} \lesssim \frac{w(B)}{|B|^{\frac{q}{q_1}}},$$

where, in the last estimate we have used that $w\in RH_{(\frac{q_1}{q})'}$. Let $a>0$ be such that $n\alpha=a(\frac{q_1}{q})'$, then by Hölder's inequality and the above inequality

$$\Big(\fint_{(2B)^c} |T_t(\chi_B\,f)|^q\,dw\Big)^{\frac{1}{q}} \lesssim \Big(\fint_{(2B)^c} |T_t(\chi_B\,f)|^{q_1}\Big(\frac{|x-x_B|}{r}\Big)^{a\frac{q_1}{q}}dx\Big)^{\frac{1}{q_1}}.$$

Now decompose $(2B)^c$ as the union of the rings $C_j(B)$ for $j\ge1$ where $C_1(B)$ is here $4B\setminus 2B$. On each $C_j(B)$ we can use the $L^{p_1}(dx)-L^{q_1}(dx)$ full off-diagonal estimates so that the right hand term is bounded by

$$\Big(\sum_{j\ge1} 2^{j\,a\,\frac{q_1}{q}}e^{-c\,4^j\,\lambda^2}\Big)^{\frac{1}{q_1}}\lambda^{\frac{n}{p_1}-\frac{n}{q_1}}\Big(\fint_B |f|^{p_1}\,dx\Big)^{\frac{1}{p_1}} \lesssim \lambda^{\frac{n}{p_1}-\frac{n}{q_1}-\frac{a}{q}}\,e^{-c\,\lambda^2}\Big(\fint_B |f|^p\,dw\Big)^{\frac{1}{p}}$$

where we have used that $w\in A_{\frac{p}{p_1}}$. Hence we have obtained (3.3) and the total power of λ is

$$\beta=\frac{n}{p_1}-\frac{n}{q_1}-\frac{a}{q}=\frac{n}{p_1}\Big(1-\frac{p}{q}\Big).$$

Let us prove the estimate (3.4). We pick p_1,q_1 as before and take a so that $a(\frac{p}{p_1})'=n\alpha'$ where $\alpha=1+(\frac{p}{p_1}-1)(\frac{q_1}{q})'$. Writing $f_j=\chi_{C_j(B)}\,f$, using that $w\in RH_{\left(\frac{q_1}{q}\right)'}$ and

by the $L^{p_1}(dx) - L^{q_1}(dx)$ full off-diagonal estimates for T_t

$$
\begin{aligned}
\Big(\fint_B |T_t(\chi_{(2B)^c}\, f)|^q\, dw\Big)^{\frac{1}{q}} &\lesssim \Big(\fint_B |T_t(\chi_{(2B)^c}\, f)|^{q_1}\, dx\Big)^{\frac{1}{q_1}} \le \sum_{j\ge1}\Big(\fint_B |T_t f_j|^{q_1}\, dx\Big)^{\frac{1}{q_1}}\\
&\lesssim \frac{1}{|B|^{\frac{1}{p_1}}}\sum_{j\ge1}\lambda^{\frac{n}{p_1}-\frac{n}{q_1}}\, e^{-c\,4^j\,\lambda^2}\Big(\int_{C_j(B)}|f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}.\\
&\lesssim \frac{1}{|B|^{\frac{1}{p_1}}}\sum_{j\ge1}\lambda^{\frac{n}{p_1}-\frac{n}{q_1}}\, e^{-c\,4^j\,\lambda^2}\, 2^{j\,\frac{a}{p_1}}\Big(\int_{C_j(B)}\Big(\frac{|x-x_B|}{r}\Big)^{-a}|f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}\\
&\le \frac{1}{|B|^{\frac{1}{p_1}}}\Big(\sum_{j\ge1}\Big[\lambda^{\frac{n}{p_1}-\frac{n}{q_1}}\, e^{-c\,4^j\,\lambda^2}\, 2^{j\,\frac{a}{p_1}}\Big]^{p_1'}\Big)^{\frac{1}{p_1'}}\Big(\sum_{j\ge1}\int_{C_j(B)}\Big(\frac{|x-x_B|}{r}\Big)^{-a}|f|^{p_1}\, dx\Big)^{\frac{1}{p_1}}\\
&\lesssim \frac{\lambda^\gamma\, e^{-c\,\lambda^2}}{|B|^{\frac{1}{p_1}}}\Big(\int_{(2B)^c}\Big(\frac{|x-x_B|}{r}\Big)^{-a}|f|^{p_1}\, dx\Big)^{\frac{1}{p_1}},
\end{aligned}
$$

where

$$\gamma = \frac{n}{p_1} - \frac{n}{q_1} - \frac{a}{p_1} = \frac{n}{q_1}\left(\frac{q}{p} - 1\right).$$

Using Hölder's inequality with $\frac{p}{p_1} > 1$ it follows that

$$
\begin{aligned}
&\Big(\fint_B |T_t(\chi_{(2B)^c}\, f)|^q\, dw\Big)^{\frac{1}{q}}\\
&\lesssim \frac{\lambda^\gamma\, e^{-c\,\lambda^2}}{|B|^{\frac{1}{p_1}}}\Big(\int_{(2B)^c}|f|^p w\, dx\Big)^{\frac{1}{p}}\Big(\int_{(2B)^c} w(x)^{1-(\frac{p}{p_1})'}\Big(\frac{|x-x_B|}{r}\Big)^{-a(\frac{p}{p_1})'}\, dx\Big)^{\frac{1}{p_1\,(\frac{p}{p_1})'}}\\
&\lesssim \lambda^\gamma\, e^{-c\,\lambda^2}\Big(\fint_{(2B)^c}|f|^p\, dw\Big)^{\frac{1}{p}}\frac{w(B)^{\frac{1}{p}}}{|B|^{\frac{1}{p_1}}}\Big(\int_{(2B)^c} w^{1-(\frac{p}{p_1})'}(x)\Big(\frac{|x-x_B|}{r}\Big)^{-n\,\alpha'}\, dx\Big)^{\frac{1}{p_1\,(\frac{p}{p_1})'}}\\
&\lesssim \lambda^\gamma\, e^{-c\,\lambda^2}\Big(\fint_{(2B)^c}|f|^p\, dw\Big)^{\frac{1}{p}},
\end{aligned}
$$

where the latter inequality is obtained as follows: since $w^{(\frac{q_1}{q})'} \in A_\alpha$, one has that $w^{(\frac{q_1}{q})'(1-\alpha')} = w^{1-(\frac{p}{p_1})'} \in A_{\alpha'}$. Thus $(viii)$ in Proposition A.1 and $w \in A_{\frac{p}{p_1}}$ imply

$$
\begin{aligned}
\Big(\int_{(2B)^c} w^{1-(\frac{p}{p_1})'}(x)\Big(\frac{|x-x_B|}{r}\Big)^{-n\alpha'}\, dx\Big)^{\frac{1}{p_1\,(\frac{p}{p_1})'}} &\lesssim \Big(\int_B w^{1-(\frac{p}{p_1})'}(x)\, dx\Big)^{\frac{1}{p_1\,(\frac{p}{p_1})'}}\\
&\lesssim |B|^{\frac{1}{p_1\,(\frac{p}{p_1})'}}\left(\frac{|B|}{w(B)}\right)^{\frac{(\frac{p}{p_1})'-1}{p_1\,(\frac{p}{p_1})'}} = \frac{|B|^{\frac{1}{p_1}}}{w(B)^{\frac{1}{p}}}.
\end{aligned}
$$

Appendix A. Muckenhoupt weights

A weight w is a non-negative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathcal{X}$,

$$\Big(\fint_B w\, d\mu\Big)\Big(\fint_B w^{1-p'}\, d\mu\Big)^{p-1} \le C.$$

For $p=1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathcal{X}$

$$\fint_B w\,d\mu \le C\,w(y), \qquad \text{for a.e. } y \in B,$$

or, equivalently, $Mw \le C\,w$ for a.e.. The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1<q<\infty$, if

$$\Big(\fint_B w^q\,d\mu\Big)^{\frac{1}{q}} \le C \fint_B w\,d\mu$$

for every ball B. The endpoint $q=\infty$ is given by the condition: $w \in RH_\infty$ whenever, for any ball B,

$$w(y) \le C \fint_B w\,d\mu, \qquad \text{for a.e. } y \in B.$$

Notice that we have excluded the case $q=1$ since the class RH_1 consists of all the weights and that is the way RH_1 is understood.

Next, we present some of the properties of these classes.

Proposition A.1.

(*i*) $A_1 \subset A_p \subset A_q$ *for* $1 \le p \le q < \infty$.

(*ii*) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \le q \le \infty$.

(*iii*) *If* $w \in A_p$, $1<p<\infty$, *then there exists* $1<q<p$ *such that* $w \in A_q$.

(*iv*) *If* $w \in RH_q$, $1<q<\infty$, *then there exists* $q<p<\infty$ *such that* $w \in RH_p$.

(*v*) $A_\infty = \displaystyle\bigcup_{1\le p<\infty} A_p = \bigcup_{1<q\le\infty} RH_q$

(*vi*) *If* $1<p<\infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.

(*vii*) *If* $1 \le q \le \infty$ *and* $1 \le s < \infty$, *then* $w \in A_q \cap RH_s$ *if and only if* $w^s \in A_{s\,(q-1)+1}$.

(*viii*) *In the case of the Euclidean space* $\mathbb{R}^n$, *if* $w \in A_p$, $1<p<\infty$, *there exists* C_w *such that for every ball* $B = B(x_B, r_B)$,

$$\int_{\mathbb{R}^n \setminus 2\,B} w(x) \left(\frac{|x-x_B|}{r_B}\right)^{-n\,p} dx \le C_w\,w(B).$$

Properties (i)-(vi) are standard. For (vii) see [JN] in the Euclidean setting (and the same argument holds in spaces of homogeneous type [ST]). The last property follows easily by using the boundedness of M on $L^p(w)$ applied to $f = \chi_B$.

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